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CHEBYSHEV NETS IN A SUBSPACE OF A GENERALIZED WEYL SPACE

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ABSTRACT

Let GW_m be m dimentional subspace of an n-dimensional subspace of an n-dimentional generalized Weyl space GW_n and let $\delta = (v_1, v_2, ..., v_m)$ be a net in GW_m. In this paper the necessary and the sufficient condition that the net δ is to be a chebyshev net and a geodesic net relative to GW_m and GW_n is obtained.
Keywords: Generalized Weyl Space, Chebyshev Net, Geodesic Net.
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GENELLEŞTİRİLMİŞ WEYL UZAYININ ALT UZAYLARINDA CHEBYSHEV ŞEBEKELERİ

ÖZET

n boyutlu GW_n genellleştirilmiş Weyl uzayının m boyutlu GW_m alt uzayına ait bir şebeke $\delta = (v_1, v_2, ..., v_m)$ olsun. Bu çalışmada δ şebekesinin GW_m ve GW_n e göre Chebyshev şebekesi ve jeodezik şebeke olması için gerek ve yeter koşullar elde edilmiştir.
Anahtar Sözcükler: Genelleştirilmiş weyl uzayı, Chebyshey şebekesi Geodesic şebeke.

1. INTRODUCTION

An m dimentional GW_m is said to be generalized Weyl space if it has an asymmetric conformal metric tensor g_{ij} and an asymmetric connection ∇_k satisfying the compatibility condition given by equation

$$\nabla_k g_{ij} = 2T_k g_{ij} \tag{1.1}$$

where T_k denotes a covariant vector field and ∇_k denotes the usual covariant derivative

Under a renormalization of the fundamental tensor of the form $\breve{g}_{ij} = \lambda^2 g_{ij}$, the complementary vector field T_k is transformed by the law $\breve{T}_k = T_k + \partial_k \ln \lambda$, where λ is a scalar function on GW_m.

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Let L^i_{jk} be denote the coefficients of the asymmetric connection ∇_k. So, a generalized Weyl space is shortly written as $GW_m(L^i_{jk}, g_{ij}, T_k)$.

The main properties of $GW_m(L^i_{jk}, g_{ij}, T_k)$ can be expressed as follows

$$g_{ij} = g_{(ij)} + g_{[ij]} \tag{1.2}$$

$$\nabla_k g_{(ij)} = 2 g_{(ij)} T_k \tag{1.3}$$

$$\nabla_k g_{[ij]} = 2 g_{[ij]} T_k \tag{1.4}$$

$$g_{(ik)} g^{(kl)} = \delta^l_i \tag{1.5}$$

$$\nabla_k g^{(ij)} = -2 T_k g^{(ij)} \tag{1.6}$$

where $g_{(ij)}$ and $g_{[ij]}$ denote symmetric and antisymmetric parts of g_{ij} respectively.

The symmetric part of the connection coefficents L^i_{jk} are given as ([1] , [2], [3], [4])

$$L^i_{(jk)} = W^i_{jk} = \begin{bmatrix} i \\ jk \end{bmatrix} - (\delta^i_j T_k + \delta^i_k T_j - g_{(jk)} g^{(mi)} T_m) \tag{1.7}$$

where $\begin{bmatrix} i \\ jk \end{bmatrix}$ are second kind Christoffel symbols defined by

$$\begin{bmatrix} i \\ jk \end{bmatrix} = \frac{1}{2} g^{(ir)} [\frac{\partial g_{(jr)}}{\partial x^k} + \frac{\partial g_{(kr)}}{\partial x^j} - \frac{\partial g_{(jk)}}{\partial x^r}] \tag{1.8}$$

A quantity A is called a satellite of weight $\{p\}$ of tensor g_{ij}, if it admits a transformation of the form

$$\breve{A} = \lambda^p A \tag{1.9}$$

The prolonged covariant derivative of a satellite A of the tensor g_{ij} of weight $\{p\}$ is defined by

$$\dot{\nabla}_k A = \nabla_k A - p T_k A \tag{1.10}$$

Let $GW_m(L^i_{jk}, g_{ij}, T_k)$ be a subspace, with coordinates u^i, of the Weyl space $GW_n(L^\alpha_{\beta\gamma}, g_{\alpha\beta}, T_\gamma)$ with coordinates x^α. Suppose that the metrics of $GW_m(L^i_{jk}, g_{ij}, T_k)$ and $GW_n(L^\alpha_{\beta\gamma}, g_{\alpha\beta}, T_\gamma)$ are elliptic and that they are given, respectively, by $g_{ij}\, du^i\, du^j$ and $g_{\alpha\beta}\, du^\alpha\, du^\beta$ which are connected by the relations

$$g_{ij} = g_{\alpha\beta}\, x^\alpha_i x^\beta_j \tag{1.11}$$

where x^α_i denotes the covariant derivative of x^α with respect to u^i.

Let $\underset{r}{v}^{i}$ $(r=1,2,...,m)$ be the contravariant components of the m independent vector fields $\underset{r}{v}$ in GW_m which are normalized by the condition $g_{ij}\underset{r}{v}^{i}\underset{r}{v}^{j}=1$. The vector fields $\underset{1}{v},\underset{2}{v},...,\underset{m}{v}$ determine a net $(\underset{1}{v},\underset{2}{v},...,\underset{m}{v})$ in GW_m.

Corresponding to the vector fields $\underset{r}{v}$; the covector fields $\overset{r}{v}$ are defined by the conditions

$$\underset{r}{v}^{i}\overset{r}{v}_{j}=\delta_{j}^{i} \quad , \quad \underset{r}{v}^{i}\overset{p}{v}_{i}=\delta_{r}^{p} \tag{1.12}$$

Denote by $\underset{\sigma}{n}^{\alpha}$ $(1,2,....,n-m)$, the contravariant components of the $n-m$ linear independent unit vector fields $\underset{\sigma}{n}$ in GW_n normal to GW_m.

The moving frame $\{x_{\alpha}^{i}\ ,\overset{\sigma}{n}_{\alpha}\}$ on GW_m, reciprocal to the moving frame $\{x_{i}^{\alpha}\ ,\underset{\sigma}{n}^{\alpha}\}$ is defined by the relations

$$\overset{\mu}{n}_{\alpha}\underset{\sigma}{n}^{\alpha}=\delta_{\sigma}^{\mu}\ ,\ \overset{\sigma}{n}_{\alpha}x_{i}^{\alpha}=0\ ,\ x_{\alpha}^{i}\underset{\sigma}{n}^{\alpha}=0\ ,\ x_{i}^{\alpha}x_{\alpha}^{j}=\delta_{i}^{j} \tag{1.13}$$
$$(\mu=1,2,...,\ n-m)$$

On the other hand, if the components of $\underset{r}{v}$ and $\overset{r}{v}$ relative to GW_n, are respectively denoted by $\underset{r}{v}^{\alpha}$ and $\overset{r}{v}_{\alpha}$, we have

$$\underset{r}{v}^{\alpha}=\underset{r}{v}^{i}x_{i}^{\alpha}\quad ,\quad \overset{r}{v}_{\alpha}=\overset{r}{v}_{i}x_{\alpha}^{i} \tag{1.14}$$

The derivation formula for $\underset{r}{v}$ and $\overset{r}{v}$ are given by [3], [5]

$$\overset{\bullet}{\nabla}_{k}\underset{r}{v}^{i}=\underset{r}{\overset{p}{T}}_{k}\underset{p}{v}^{i}\quad ,\quad \overset{\bullet}{\nabla}_{k}\overset{r}{v}_{i}=-\underset{p}{\overset{r}{T}}_{k}\overset{p}{v}_{i} \tag{1.15}$$

Taking the prolonged covariant derivative of $(1.13)_4$ with respect to u^k and remembering that [5]

$$\overset{\bullet}{\nabla}_{k}x_{i}^{\alpha}=\sum_{\sigma=1}^{n-m}\overset{\sigma}{w}_{ik}\underset{\sigma}{n}^{\alpha}+A_{ik}^{h}x_{h}^{\alpha} \tag{1.16}$$

where $\overset{\sigma}{w}_{ik}$ are the components of second fundemental form related to the normal $\overset{\sigma}{n}$ of GW_m defined by

$$\overset{\sigma}{w}_{ik}=\overset{\sigma}{n}_{\alpha}\overset{\bullet}{\nabla}_{k}x_{i}^{\alpha} \tag{1.17}$$

and A^h_{ik} are defined by

$$A^h_{ik} = x^h_\alpha \overset{\bullet}{\nabla}_k x^\alpha_i \tag{1.18}$$

From $(1.16)_4$, we have

$$\nabla_k x^j_\alpha = \sum_{\sigma=1}^{n-m} \underset{\sigma}{\Omega}{}^j_k \overset{\sigma}{n}_\alpha - A^j_{hk}\, x^h_\alpha \quad , \quad \underset{\sigma}{\Omega}{}^i_k = n^\alpha \nabla_k x^j_\alpha \tag{1.19}$$

For the definition of the Chebyshev nets of the first and second kind, and that the b-nets and c-nets, the reader is refereed to [3].

2. CHEBYSHEV NETS IN A GENERALIZED WEYL SUBSPACE

Consider the net ($\underset{1}{v}, \underset{2}{v}, ..., \underset{m}{v}$) in GW_m determined by the vector fields $\underset{1}{v}, \underset{2}{v}, ..., \underset{m}{v}$. Let $\underset{rp}{\overline{a}}{}^\alpha$, $\underset{rp}{a}{}^i$; $\overset{r}{\overline{b}}_\alpha$, $\overset{r}{b}_i$ and $\underset{r}{\overline{c}}{}^\alpha$, $\underset{r}{c}{}^i$; be the covariant components of the chebyshev vector fields of the first and the second kind and the geodesic vector fields of the given net relative to GW_n and GW_m, respectively.

Theorem 2.1. For the Chebyshev vector fields of the first and second kind and the geodesic vector fields of the net ($\underset{1}{v}, \underset{2}{v}, ..., \underset{m}{v}$) in GW_m we have

(i) $$\underset{rp}{\overline{a}}{}^\alpha = \underset{p}{v}{}^\gamma \overset{\bullet}{\nabla}_\gamma A = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{p}{v}{}^k \underset{r}{v}{}^i \underset{\sigma}{n}{}^\alpha + \underset{rp}{a}{}^h x^\alpha_h + A^h_{ik} \underset{r}{v}{}^i \underset{p}{v}{}^k x^\alpha_h$$

$$\overset{\sigma}{w}_{ik} = g_{(\alpha\beta)} \underset{\sigma}{n}{}^\beta \overset{\bullet}{\nabla}_k x^\alpha_i \qquad (r \neq p)$$

(ii) $$\overset{®}{\overline{b}}_\alpha = \underset{®}{v}{}^\gamma \overset{\bullet}{\nabla}_\gamma \overset{®}{v}_\alpha = -\sum_{\sigma=1}^{n-m} \Omega^i_k \overset{®}{v}_i \underset{®}{v}{}^k \overset{\sigma}{n}_\alpha + \overset{r}{b}_h x^h_\alpha + A^i_{hk} x^h_\alpha \overset{®}{v}_i \underset{®}{v}{}^k$$

$$\underset{\sigma}{\Omega}{}^j_k = \underset{\sigma}{n}{}^\alpha \overset{\bullet}{\nabla}_k x^i_\alpha$$

(iii) $$\underset{r}{\overline{c}}{}^\alpha = \underset{r}{v}{}^\gamma \overset{\bullet}{\nabla}_\gamma v^\alpha_r = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}{}^i \underset{r}{v}{}^k \underset{\sigma}{n}{}^\alpha + x^\alpha_h c^h_r + A^h_{ik} \underset{r}{v}{}^i \underset{r}{v}{}^k x^\alpha_h$$

where the symbol ® indicates that the summation on R will not be made.

Proof. (i) Taking the prolonged covariant derivative of each side of $(1.14)_1$ with respect to u^k and using (1.15) and (1.16), we get

$$\overset{\bullet}{\nabla}_k \underset{r}{v}{}^\alpha = \underset{r}{v}{}^i (\overset{\bullet}{\nabla}_k x^\alpha_i) + x^\alpha_i (\overset{\bullet}{\nabla}_k \underset{r}{v}{}^i) = \underset{r}{v}{}^i (\sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} n^\alpha_\sigma + A^h_{ik} x^\alpha_h) + x^\alpha_i (\underset{r}{\overset{p}{T_k}} \underset{p}{v}{}^i) \tag{2.1}$$

$$\underset{l}{v}^k \overset{\bullet}{\nabla}_k \underset{r}{v}^\alpha = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}^i \underset{l}{v}^k n_\sigma^\alpha + A_{ik}^h \underset{r}{v}^i \underset{l}{v}^k x_h^\alpha + x_i^\alpha (\underset{r}{\overset{p}{T}}_k \underset{l}{v}^k) \underset{p}{v}^i \tag{2.2}$$

In view of

$$\underset{l}{v}^k \overset{\bullet}{\nabla}_k = \underset{l}{v}^\gamma \overset{\bullet}{\nabla}_\gamma \quad , \quad \underset{r}{\overset{p}{T}}_k \underset{l}{v}^k = \underset{rl}{\overset{p}{\tau}} \tag{2.3}$$

the last relation becomes

$$\underset{l}{v}^k \overset{\bullet}{\nabla}_\gamma \underset{r}{v}^\alpha = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}^i \underset{l}{v}^k \underset{\sigma}{n}^\alpha + A_{ik}^h \underset{r}{v}^i \underset{l}{v}^k x_h^\alpha + x_i^\alpha (\underset{rl}{\overset{p}{\tau}} \underset{p}{v}^i) \tag{2.4}$$

where the functions $\underset{rl}{\overset{p}{\tau}}$ $(r \neq l)$ are the chebyshev curvatures of the first kind of the curves of the net

$$\underset{rp}{a} = \underset{rp}{\overset{s}{\tau}} \underset{s}{v}^i \tag{2.5}$$

So that (2.4) reduces to

$$\underset{rl}{\bar{a}}^\alpha = \underset{l}{v}^\gamma \overset{\bullet}{\nabla}_\gamma A = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{l}{v}^k \underset{r}{v}^i \underset{\sigma}{n}^\alpha + \underset{rl}{a}^h x_h^\alpha + A_{ik}^h \underset{r}{v}^i \underset{l}{v}^k x_h^\alpha \qquad (r \neq l) \tag{2.6}$$

(ii) Differentiating covariantly both sides of $(1.14)_2$ with respect to u^k and taking (1.19) and $(1.15)_2$ into consideration we obtain

$$\begin{aligned} \overset{\bullet}{\nabla}_k \overset{r}{v}_\alpha &= \overset{\bullet}{\nabla}_k (\overset{r}{v}_i x_\alpha^i) = (\overset{\bullet}{\nabla}_k \overset{r}{v}_i) x_\alpha^i + \overset{r}{v}_i (\overset{\bullet}{\nabla}_k x_\alpha^i) \\ &= (\sum_{\sigma=1}^{n-m} \underset{\sigma}{\Omega}_k^i \overset{\sigma}{n}_\alpha - A_{hk}^i x_\alpha^h) \overset{r}{v}_i - \underset{p}{\overset{r}{T}}_k \overset{p}{v}_i x_\alpha^i \end{aligned} \tag{2.7}$$

Multiplying (2.7) by $\underset{\circledR}{v}^k$ and remembering that $\overset{\bullet}{\nabla}_k \overset{r}{v}_\alpha = x_k^\gamma \overset{\bullet}{\nabla}_\gamma \overset{r}{v}_\alpha$, we have

$$\underset{\circledR}{v}^\gamma \overset{\bullet}{\nabla}_\gamma \overset{\circledR}{v}_\alpha = \sum_{\sigma=1}^{n-m} \underset{\sigma}{\Omega}_k^i \overset{\sigma}{n}_\alpha \overset{\circledR}{v}_i \underset{\circledR}{v}^k - A_{hk}^i \overset{\circledR}{v}_i \underset{\circledR}{v}^k x_\alpha^h - \underset{p}{\overset{r}{T}}_k \overset{p}{v}_i \underset{r}{v}^k x_\alpha^i \tag{2.8}$$

Since the Chebyshev curvatures $\underset{l}{\overset{r}{\rho}}$ of the second kind of the curves belonging to the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{m}{v})$ are defined by

$$\underset{l}{\overset{r}{\rho}} = \underset{p}{\overset{\circledR}{T}}_k \underset{\circledR}{v}^k \tag{2.9}$$

The last relation takes the form

$$\underset{\circledR}{v}^\gamma \overset{\bullet}{\nabla}_\gamma \overset{\circledR}{v}_\alpha = \sum_{\sigma=1}^{n-m} \underset{\sigma}{\Omega}_k^i \overset{\sigma}{n}_\alpha \overset{\circledR}{v}_i \underset{\circledR}{v}^k - A_{hk}^i \overset{\circledR}{v}_i \underset{\circledR}{v}^k x_\alpha^h - (\underset{l}{\overset{r}{\rho}} \overset{l}{v}_i) x_\alpha^i \tag{2.10}$$

or, in terms of the chebyshev vector fields of the second kind, (2.10) becomes

$$\overset{r}{b}_{\alpha} = -\sum_{\sigma=1}^{n-m} \Omega_{k}^{i} \underset{\sigma}{v_i} \overset{®}{v}{}^{k}_{®} \overset{\sigma}{n}_{\alpha} - b_{h}^{r} x_{\alpha}^{h} - A_{hk}^{i} x_{\alpha}^{h} \overset{®}{v_i} \underset{®}{v}{}^{k} \tag{2.11}$$

where $\overset{r}{b}_i$ is defined by $\overset{r}{b}_i = \overset{r}{\underset{s}{\rho}} \overset{s}{v}_i$.

(iii) If we multiply (2.1) by $\underset{r}{v}{}^{k}$ and sum for k , we find

$$\underset{r}{v}{}^{k} \overset{\bullet}{\nabla}_{k} \underset{r}{v}{}^{\alpha} = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} \underset{\sigma}{n}{}^{\alpha} + (\overset{p}{\underset{r}{T}}_{k} \underset{r}{v}{}^{k}) \underset{p}{v}{}^{i} x_{i}^{\alpha} + A_{ik}^{h} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} x_{h}^{\alpha} \tag{2.12}$$

or using $(1.14)_1$ and the fact that $\overset{\bullet}{\nabla}_k = x_k^{\gamma} \overset{\bullet}{\nabla}_{\gamma}$, we transform the last equation into

$$\underset{r}{v}{}^{\gamma} \overset{\bullet}{\nabla}_{\gamma} \underset{r}{v}{}^{\alpha} = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} n^{\alpha} + \overset{p}{\underset{r}{S}} \underset{p}{v}{}^{i} x_{i}^{\alpha} + A_{ik}^{h} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} x_{h}^{\alpha} \tag{2.13}$$

where $\overset{p}{\underset{r}{S}}$ are geodesic curvatures of the curves belonging to the net considered and are defined by $\overset{p}{\underset{r}{S}} = \overset{p}{\underset{r}{T}}_{k} \underset{r}{v}{}^{k}$.

In terms of the geodesic vector fields of the net, (2.13) reduces to

$$\underset{r}{\overline{c}}{}^{\alpha} = \sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} n^{\alpha} + x_{h}^{\alpha} c_{r}^{h} + A_{ik}^{h} \underset{r}{v}{}^{i} \underset{r}{v}{}^{k} x_{h}^{\alpha} \tag{2.14}$$

where $\underset{r}{c}{}^{i}$ is the geodesic vector field, relative to GW_m , of the $r-$th family of the net defined by $\underset{r}{c}{}^{i} = \overset{p}{\underset{r}{S}} \underset{p}{v}{}^{i}$.

The relations (2.6), (2.11), (2.14), allow us to state the following theorem:

Theorem 2.2. (i) A neccessary and the sufficient condition the net $\delta = (\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ in GW_m which is a chebyshev net of the first kind with respect to GW_n to be a chebyshev net of the first kind with respect to GW_m is that

$$\sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{l}{v}{}^{i} \underset{l}{v}{}^{i} = 0 \quad \text{and} \quad A_{ik}^{h} \underset{r}{v}{}^{i} \underset{l}{v}{}^{k} = 0$$

(ii) A neccessary and the sufficient condition the net $\delta = (\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ in GW_m which is a chebyshev net of the second kind with respect to GW_n to be a chebyshev net of the second kind with respect to GW_m is that

$$\sum_{\sigma=1}^{n-m} \Omega_{k}^{i} \underset{\sigma}{v_i} \overset{®}{v}{}^{k}_{®} = 0 \quad \text{and} \quad A_{hk}^{i} \overset{®}{v_i} \underset{®}{v}{}^{k} = 0$$

(iii) A neccessary and the sufficient condition the net $\delta = (\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ in GW_m which is a geodesic net relative GW_n to be a geodesic net relative to GW_m is that

$$\sum_{\sigma=1}^{n-m} \overset{\sigma}{w}_{ik} \underset{r}{v}^i \underset{r}{v}^k = 0 \quad \text{and} \quad A_{ik}^h \underset{r}{v}^i \underset{r}{v}^k = 0$$

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