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VIBRATION OF VISCOELASTIC BEAMS SUBJECTED TO MOVING HARMONIC LOADS

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ABSTRACT

The transverse vibration of a beam with intermediate point constraints subjected to a moving harmonic load is analyzed within the framework of the Bernoulli-Euler beam theory. The Lagrange equations are used for examining the dynamic response of beams subjected to the moving harmonic load. The constraint conditions of supports are taken into account by using Lagrange multipliers. In the study, for applying the Lagrange equations, trial function denoting the deflection of the beam is expressed in the polynomial form. By using the Lagrange equations, the problem is reduced to the solution of a system of algebraic equations. The system of algebraic equations is solved by using the direct time integration method of Newmark [8]. Results of numerical simulations are presented for various combinations of constant axial velocity, excitation frequency, number of point supports and various values of damping coefficient.
Keywords: Forced vibrations of beam, free vibrations of beam, moving harmonic load

HAREKETLİ HARMONİK YÜKLER ETKİSİNDEKİ VİSKOELASTİK KİRİŞLERİN TİTREŞİMİ

ÖZET

Bu çalışmada hareketli harmonik yükler etkisindeki kirişlerin enine titreşimleri Bernoulli-Euler kiriş teorisi çerçevesinde incelenmiştir. Problemin çözümü için Lagrange denklemleri kullanılmıştır. Problemde mesnet şartları Lagrange çarpanları kullanılarak sağlanmıştır. Çalışmada, Lagrange denklemlerinin uygulanması için kirişin yerdeğiştirmelerini ifade eden çözüm fonksiyonunun oluşturulmasında polinomlar kullanılmıştır. Lagrange denklemleri kullanılarak problem cebrik denklem sisteminin çözümüne indirgenmiştir. Bu denklem sistemi Newmark [8] yöntemi kullanılarak çözülmüştür. Problemde kirişin yerdeğiştirmeleri, hareketli harmonik yükün frekansı ve hızı, çeşitli sönüm oranları ve açıklık sayısı için sayısal olarak incelenmiştir.
Anahtar Sözcükler: Zorlanmış kiriş titreşimleri, serbest kiriş titreşimleri, hareketli harmonik yük

1. INTRODUCTION

Transverse vibration of beams subjected to moving loads has been an interesting research topic for long years. Vibrations of this kind occur in many branches of engineering, for example in bridges and railways. Many methods have been presented for response prediction, but only the notable ones cited here. The earliest work on the the behaviour of a single-span beam subjected to a constant moving harmonic load was reported by Timoshenko and Young [1]. Fryba [2] presented various analytical solutions for vibration problems of simple and continuous beams under moving loads in his book. H.P. Lee [3] utilized Hamilton's principle to solve the dynamic response of a beam with intermediate point constraints subjected to a moving load by using the

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vibration modes of a simply supported beam as the assumed modes. Abu-Hilal and Mohsen [4] studied the dynamic response of elastic homogenous isotropic beams with different boundary conditions subjected to a constant force travelling with accelerating, decelerating and constant velocity types of motion. Zheng et al. [5] considered the dynamic response of the continuous beams subjected to moving loads by using the modified beam vibration functions. Dugush and Eisenberg [6] examined vibrations of non-uniform continuous beams under moving loads by using both the modal analysis method and the direct integration method. Zhu and Law [7] analyzed the dynamic response of a continuous beam under moving loads using Hamilton's principle and eigenpairs obtained by the Ritz method.

In the present study, the Lagrange equations are used for examining the dynamic response of viscoelastic beams subjected to a moving harmonic load with constant axial speed. The constraint conditions of the supports are taken into account by using Lagrange multipliers. In the study, for applying the Lagrange equations, the trial function denoting the deflection of the beam is expressed in the polynomial form. By using the Lagrange equations, the problem is reduced to a system of algebraic equations. This system of algebraic equations is solved by using the direct time integration method of Newmark [8]. The convergence of the study is based on the numerical values obtained for various numbers of polynomial terms. Results in this paper are readily applicable for further investigation in this field.

2. THEORY AND FORMULATIONS

A continuous viscoelastic Bernoulli-Euler beam with N point supports subjected to a moving harmonic load is depicted in Fig 1. The considered beam has a uniform cross sectional area, and its length is L. The beam is constrained against vertical displacements at various points. The constraint conditions are satisfied by using Lagrange multipliers. A moving harmonic load $Q(t)$ is applied in the *y* direction from left to right with prescribed constant speed in the axial direction. The assumptions made in the following formulation are that transverse deflections are small so that the dynamic behaviour of the beam is governed by the Bernoulli-Euler beam theory. Moreover, all the transverse deflections occur in the same plane, defined by the *x* and *y* axes. The *y* axis is chosen at the midpoint of the total length of the beam as shown in Fig. 1.

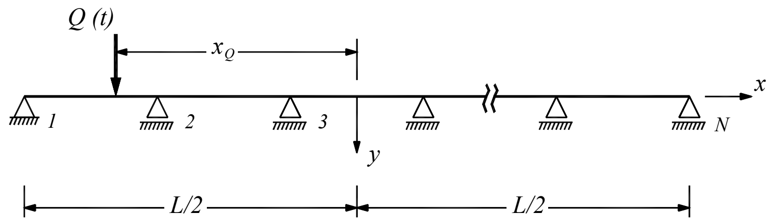

Figure 1. A continuous beam with *N* point supports subjected to a moving harmonic load

According to the Bernoulli-Euler beam theory, the elastic strain energy of the beam at any time in Cartesian coordinates due to bending is

$$U = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} E\,I(x)\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 \mathrm{d}x \qquad (1)$$

where E, $I(x)$ and $w(x,t)$ are the Young's modulus and the moment of inertia of the cross section of the beam and the displacement function of the beam.

Neglecting the rotatory inertia effects, the kinetic energy of the beam at any time is

$$T = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho A(x) \left(\frac{\partial w(x,t)}{\partial t}\right)^2 \mathrm{d}x \tag{2}$$

where ρ and $A(x)$ are the mass density and the cross-section area of the beam. The Kelvin model for the material is used. In this case, the dissipation function of the beam at any time is

$$R = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} r_i \left(\frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\right)^2 \mathrm{d}x \tag{3}$$

$$r_i = \gamma_2 E\, I(x) \tag{4}$$

where r_i and γ_2 are the coefficient of internal damping of the viscoelastic beam and proportionality constant of internal damping, respectively. The potential energy of the external force $Q(t)$ at any time is

$$V = -Q(t)\, w\left(x_Q, t\right) \tag{5a}$$

$$Q(t) = P\sin(\Omega t) \tag{5b}$$

where P is the amplitude of the moving harmonic force, Ω is the excitation frequency, x_Q is the coordinate of the moving harmonic load at any time t and expressed as

$$x_Q = v.t - L/2 \;;\quad -\frac{L}{2} \le x_Q \le \frac{L}{2}\;;\qquad 0 \le t \le \frac{L}{v} \tag{6}$$

where v is the axial velocity of the moving harmonic load. The functional of the problem is

$$I = T - (U + V) \tag{7}$$

It is known that some expressions satisfying geometrical boundary conditions are chosen for $w(x,t)$ and by using the Lagrange equations, the natural boundary conditions are also satisfied. Therefore, by using the Lagrange equations and by assuming the displacement $w(x,t)$ to be representable by a linear series of admissible functions and adjusting the coefficients in the series to satisfy the Lagrange equations, an approximate solution is found for the displacement function. For applying the Lagrange equations, the trial function $w(x,t)$ is approximated by space-dependent polynomial terms $x^0, x^1, x^2, \ldots, x^M$ and time-dependent generalized displacement coordinates $A_m(t)$. Thus

$$w(x,t) = \sum_{m=0}^{M} A_m(t)\, x^m \tag{8}$$

where $w(x,t)$ is the dynamic response of the beam subjected to the moving harmonic load. The constraint conditions of the supports are satisfied by using the Lagrange multipliers. The constraint conditions are

$$\lambda_i\, w\left(x_{Si}, t\right) = 0\,, \qquad\qquad i = 1, 2, 3, \ldots\ldots, N \tag{9}$$

where $x_{S\,i}$ denotes the location of the i th support, N denotes the number of the point supports. In Eq. (9), λ_i quantities are the Lagrange multipliers which are the support reactions in the considered problem. The Lagrange multipliers formulation of the considered problem necessities the construction of the Lagrangian functional;

$$L = I + \lambda_i \, w\left(x_{S\,i}, t\right), \qquad i = 1, 2, 3,\ldots, N \tag{10}$$

which attains its stationary value at the solution $\left(w(x_{Si}, t), \lambda_i\right)$. The generalized damping force Q_{D_r} can be obtained from the dissipation function by differentiating R with respect to $\dot{A}_k$

$$Q_{D_r} = -\frac{\partial R}{\partial \dot{A}_k}, \qquad k = 1, 2, 3,\ldots, M{+}N \tag{11}$$

Then, using the Lagrange equations

$$\frac{\partial L}{\partial A_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{A}_k} + Q_{D_r} = 0, \qquad k = 1, 2, 3,\ldots, M{+}N \tag{12}$$

where the dot above is the derivative with respect to time, and introducing

$$A_{M+i} = \lambda_i, \qquad i = 1, 2, 3,\ldots, N \tag{13}$$

yield the following equation;

$$[A]\{A_m\} + [B]\{\dot{A}_m\} + [C]\{\ddot{A}_m\} = \{D\}, \qquad m = 1, 2, 3\ldots, M{+}N \tag{14}$$

Where

$$A_{km} = \int_{-\frac{L}{2}}^{\frac{L}{2}} E\,I(x)(x^k)''(x^m)''\,\mathrm{d}\,x, \qquad k,m = 1, 2, 3\ldots, M$$

$$A_{km} = x_{S\,m}^{k}, \qquad k = 1, \ldots, M; \quad m = M, \ldots, M+N$$

$$A_{km} = x_{S\,k}^{m}, \qquad k = M, \ldots, M+N; \quad m = 1, \ldots, M$$

$$A_{km} = 0, \qquad k,m = M, \ldots, M+N$$

$$B_{km} = \int_{-\frac{L}{2}}^{\frac{L}{2}} r_i\,(x^k)''(x^m)''\,\mathrm{d}\,x, \qquad k,m = 1, 2, 3\ldots, M$$

$$B_{km} = 0, \qquad k,m = M, \ldots, M+N$$

$$C_{km} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho\,A(x) x^k x^m\,\mathrm{d}\,x, \qquad k,m = 1, 2, 3\ldots, M$$

$$C_{km} = 0, \qquad k,m = M, \ldots, M+N$$

$$D_k = Q\,x_Q^k, \qquad k = 1, 2, 3\ldots, M$$

$$D_k = 0, \qquad k = M, \ldots, M+N \tag{15}$$

where $(x^k)''$ is the second derivative of the x^k . $[A]$, $[B]$, $[C]$ are the matrices that do not depend on time, but $\{D\}$ depends on time; namely x_Q^k depens on time.

For free vibration analysis, the time-dependent generalized displacement coordinates can be expressed as follows:

$$A_m(t) = \overline{A}_m e^{i \omega t} \tag{16}$$

By substituting Eq. (16) into Eq. (14) and taking the damping matrix of the beam $[B]$ and the external forces matrix $\{D\}$ as zero in Eq. (14), this situation results in a set of linear homogeneous equations that can be expressed in the following matrix form;

$$[A]\{\overline{A}_m\} - \omega^2 [C]\{\overline{A}_m\} = \{0\} \tag{17}$$

where ω is the natural frequency of the considered beam. By using the direct time integration method of Newmark [8], Eq. (14) is solved and A_m , $\dot{A}_m$, $\ddot{A}_m$ and λ_i coefficients are obtained for any time t . Then, the displacements, velocities and accelerations at the considered point and time can be determined by using Eq. (8).

3. NUMERICAL RESULTS

A number of numerical examples are presented to demonstrate the versatility, accuracy and efficiency of the present method. The obtained results are in good agreement with the previously published results where applicable. In the following figures, x_Q is the distance between the moving harmonic load and the midpoint of the beam.

At this stage, a convergence study is carried out. For this purpose, the natural frequencies of the considered beam are determined by calculating the eigenvalues ω_i of the frequency Eq. (17). In Table 1, the calculated natural frequencies are compared with those of Timoshenko and Young [1] and Fyrba [2]. The convergence is tested by taking the number of the polynomial terms $4, 5, 6, 8, 10, 12, 14$. It is seen that the present converged values show excellent agreement with those of Timoshenko and Young [1] and Fyrba [2].

Table 1. Convergence study of the natural frequencies ω_i (rad/s) of the beam and comparison of the obtained results with the existing exact results

	Number of Polynomial Terms	ω_1	ω_2	ω_3	ω_4
Present Study	4	542.0610	2485.9376	-	-
	5	488.8704	2485.9376	6526.4929	-
	6	488.8704	1962.7350	6526.4929	13667.579
	8	488.8704	1954.9982	4473.3338	8149.2914
	10	488.8704	1954.9982	4399.8340	7830.1475
	12	488.8704	1954.9982	4397.8998	7819.5093
	14	488.8704	1954.9982	4397.8998	7819.5093
References [1] and [2]		488.6999	1954.7999	4398.2999	7819.1998

It is observed from Table 1 that, the natural frequencies decrease as the number of the polynomial terms increases: It means that the convergence to the exact value is from above. By increasing the number of the polynomial terms, the exact value can be approached from above. It should be remembered that energy methods always overestimate the fundamental frequency, so with more refined analyses, the exact value can be approached from above. Convergence study indicates that the calculated values are converged to within three significant figures.

In all of the calculations, the damping ratio ξ is taken as $0\cdot 0, 0\cdot 05, 0\cdot 10$ and dimensionless damping coefficient which is given in reference [4] as follows is considered.

$$\xi = \frac{\gamma_1 + \gamma_2 \omega_k^2}{2\omega_k} \tag{18}$$

In Eq. (18), γ_1 is the proportionality constant of the external damping, γ_2 is the proportionality constant of the internal damping and ω_k is the natural circular frequency of the k th mode, respectively. It is known that external damping is very small with respect to internal damping. Therefore, the external damping is ignored in this study. In the case of damping by using Eqs. (4) and (18), the damping coefficient r_i is obtained. From here on, the number of the polynomial terms is taken as 12 in all of the numerical investigations.

3.1. A Single-Span Beam

A single-span beam with simply-supported ends is considered. The cross sectional area A and the mass density ρ are $1\cdot 146\times 10^{-3}$ m^2 and 7700 kg /m^3, respectively. The total length L is 1 m and Young's modulus E is 207 000 MPa ($EI = 21635\cdot 18$ Nm^2). The deflection at the centre of the span due to moving harmonic load w is normalized by the static deflection D $(D = PL^3/48EI)$. The numerical integration is performed using Gaussian quadrature. The frequency ratio β is taken as defined in reference [4] as follows

$$\beta = \frac{\Omega}{\omega_1} \tag{19}$$

where ω_1 is the natural frequency at the first mode of vibration of a simply supported beam calculated from the Eq. (17), has a value of $488\cdot 87$ rad/s. The effect of the damping is represented by the damping ratio $\xi = 0\cdot 0,\ 0\cdot 05,\ 0\cdot 10$ in the calculations.

In Figs. 2-5, the normalized (w/D) deflections at the center of a single-span beam are shown. These deflections occur due to moving harmonic load travelling at constant axial velocity, $v = 15\cdot 5$, $v = 39$, $v = 78$ and $v = 155$ m/s, for various values of ξ. These results are compared with those given in references [1,3,4] for moving load and moving harmonic load. Good agreement is observed.

Figs. 2-5 show the effects of the velocity, the excitation frequency and damping for the single-span beam. In all these figures, the effect of damping is clear for all cases where an increase in damping yields, in general, a decrease in the response. It is seen from the Figs. 2-5 that for the small values of velocity, the excitation frequency has more important effect on the behaviour of the beam, i.e; for especially $v = 15\cdot 5$ m/s. The maximum dimensionless absolute displacement of the beam is increased by increasing the values of β until $\beta = 1$. The above mentioned displacement reaches a maximum value at resonance ($\beta = 1$), and then with the increase in β, it decreases. The case of resonance is much more visible for this velocity. Because, as the values of velocity increases, the acting time of the load on the beam becomes shorter. Therefore, at high values of the velocity, the load leaves the beam without completing its one-period. It is seen from the obtained results that in the case of $\beta = 0$, the maximum dynamic deflection at the centre of the span is associated with speed $v = 78$ m/s. Moreover, it should also be pointed out that negative displacement means tension stresses at the top of the beam. The dimensionless deflection for a single-span beam is independent of the magnitude of the force. In the case of moving load, namely $\beta = 0$, the obtained results are excellent agreement with those of Lee [3].

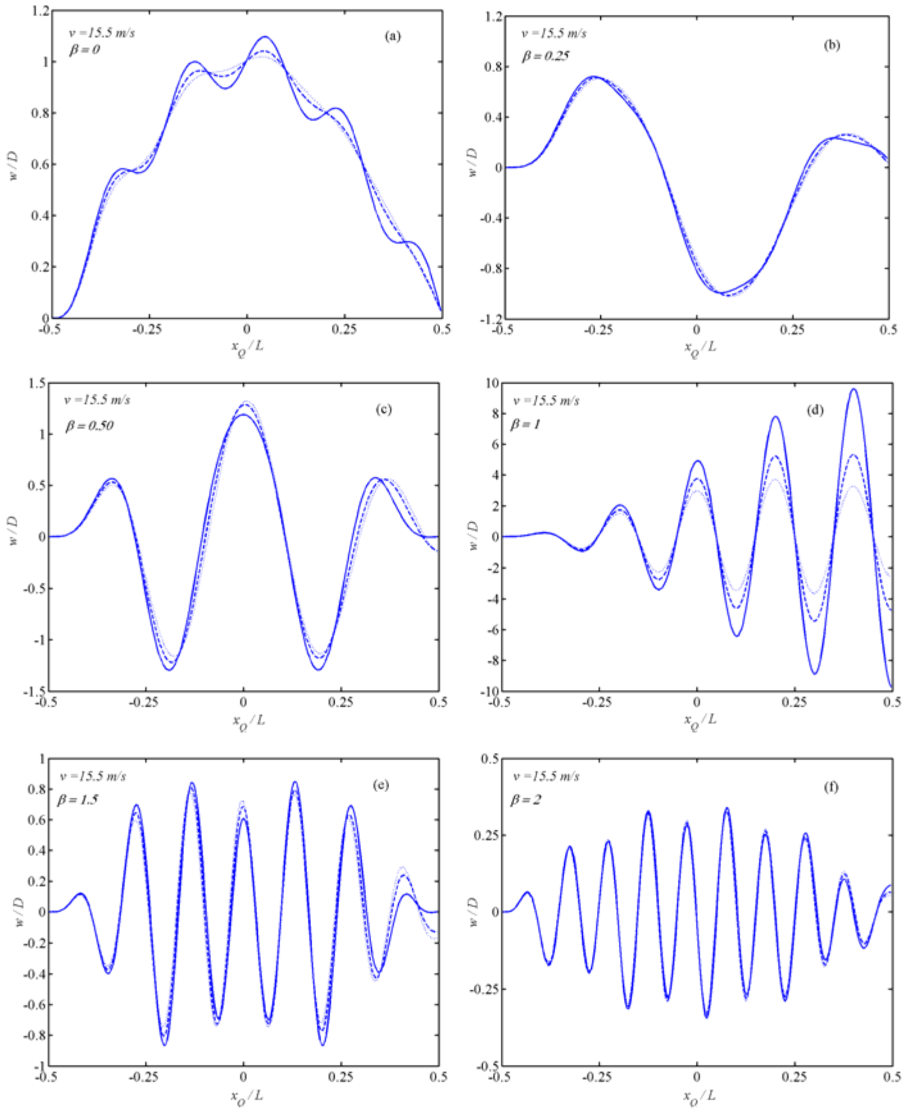

Figure 2. Normalized deflections at the centre of a single-span beam varying β for $v = 15 \cdot 5$ m/s, (——) $\xi = 0 \cdot 0$, (– – –) $\xi = 0 \cdot 05$, (·····) $\xi = 0 \cdot 10$

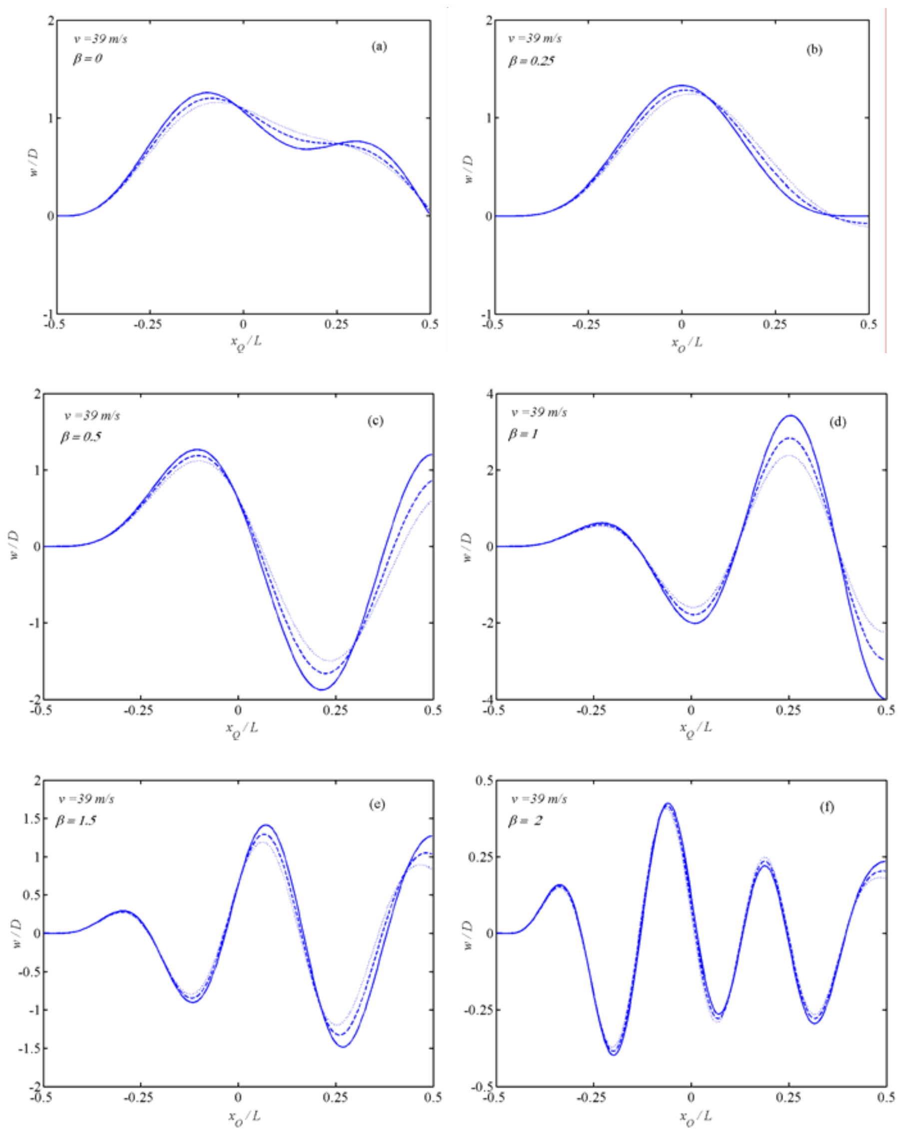

Figure 3. Normalized deflections at the centre of a single-span beam varying β for $v = 39$ m/s, (——) $\xi = 0\cdot 0$, (− − −) $\xi = 0\cdot 05$, (·····) $\xi = 0\cdot 10$

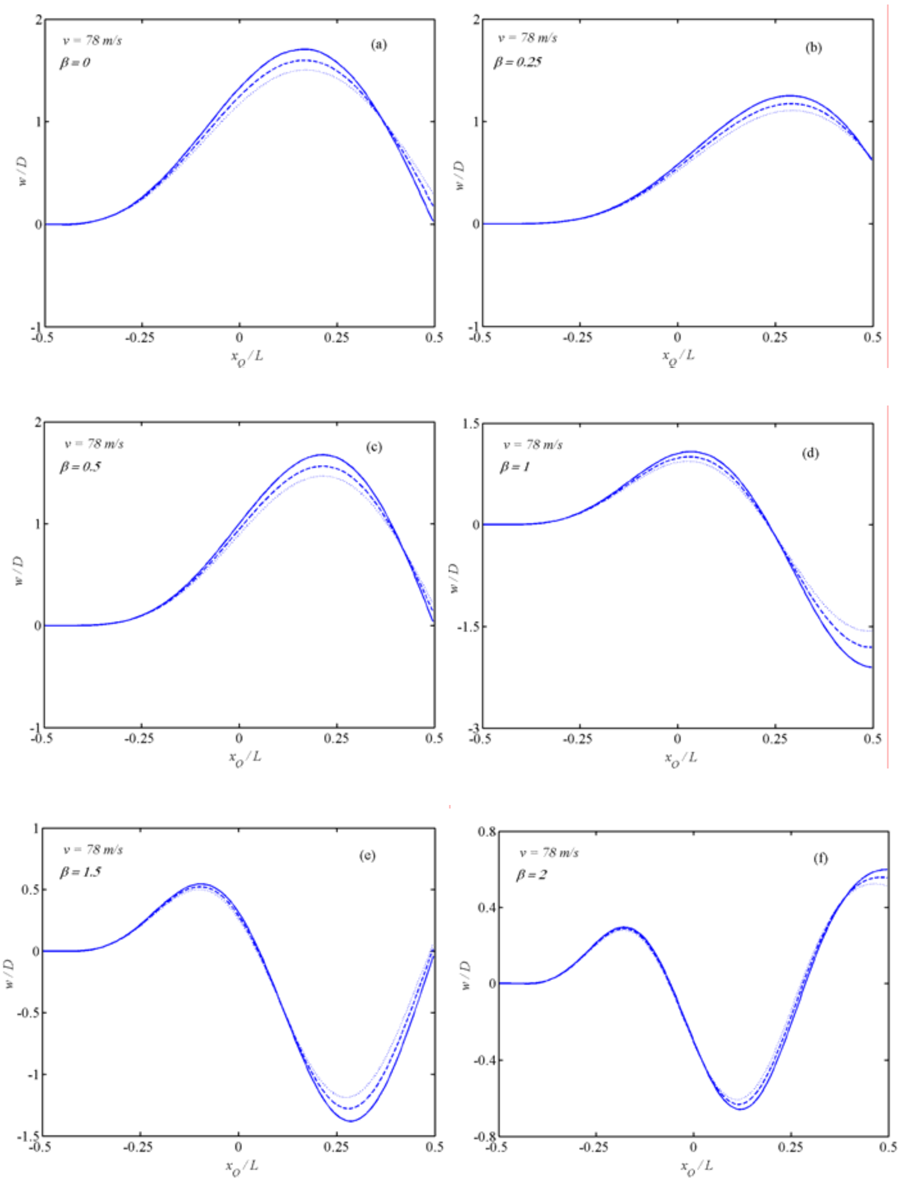

Figure 4. Normalized deflections at the centre of a single-span beam varying β for

$v = 78$ m/s, (——) $\xi = 0 \cdot 0$, $(---)$ $\xi = 0 \cdot 05$, $(\cdots\cdot)$ $\xi = 0 \cdot 10$

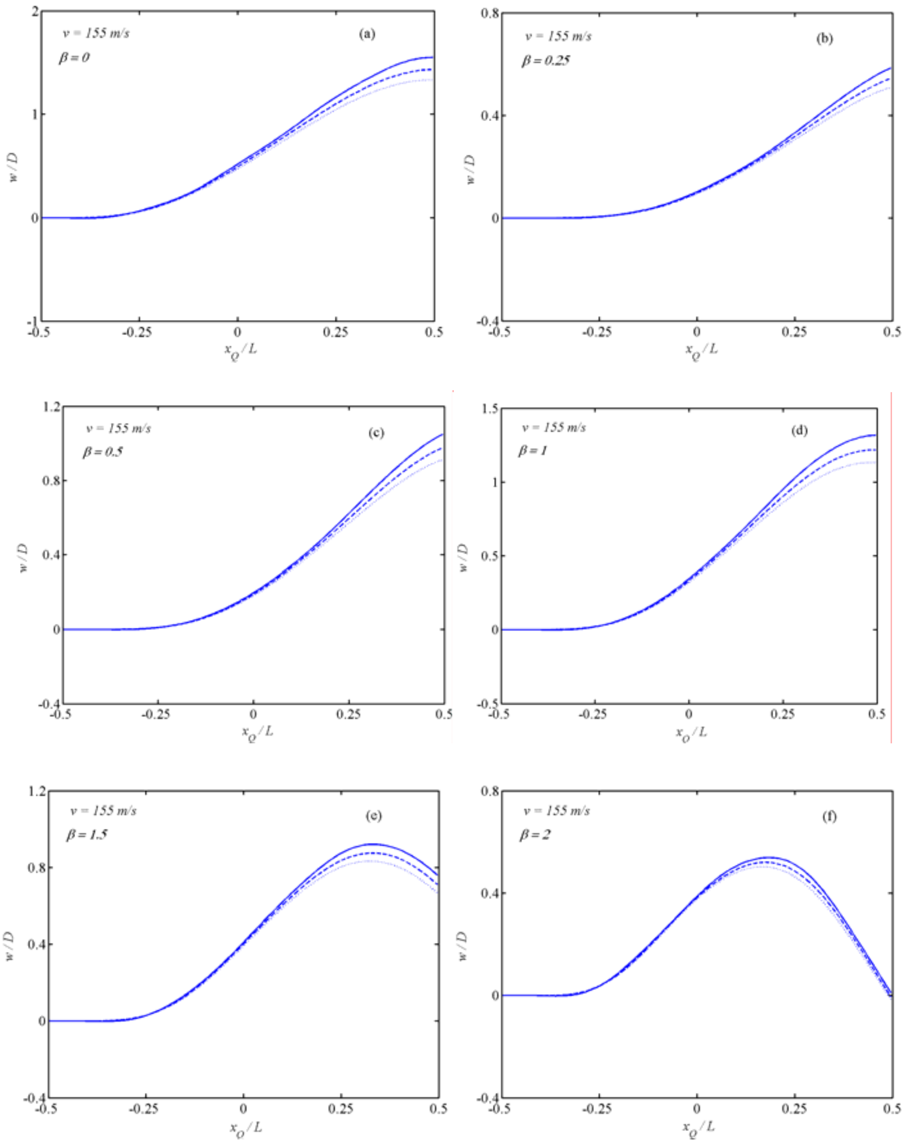

Figure 5. Normalized deflections at the centre of a single-span beam varying β for $v = 155$ m/s, (——) $\xi = 0 \cdot 0$, (− − −) $\xi = 0 \cdot 05$, (·····) $\xi = 0 \cdot 10$

3.2. Two-Span Beam

A two-span viscoelastic beam with simply-supported ends and a simple support at the middle of the span of the beam is considered. The geometrical and the physical properties of the beam are the same of those of the first problem's. The fundemental natural frequency of the beam at the first mode is $1954 \cdot 99$ rad/s which is obtained from the equation [17]. The damping ratio is assumed to be $\xi = 0 \cdot 05$. The deflections are normalized by the deflection D $(D = PL^3 / 48EI)$ of the single-span beam. Figs. 6-7 show the normalized deflections under the moving harmonic load for a two-span viscoelastic beam. In the case of moving load, namely $\beta = 0$, the present results are in perfect agreement with those given in references [3] and [5].

Fig. 6 shows the dynamic response of a two-span beam for different values of the axial velocity of the load for $\beta = 1$ and $\xi = 0 \cdot 05$. For the considered parameters, it is noticed that in the low velocities of moving harmonic load, i.e., $v = 15 \cdot 5$ m/s, the beam has much more higher maximum dynamic displacement than in the fast velocities of the moving harmonic load and there is a drastic decrease in the dynamic displacement by increasing the value of the velocity.

Fig. 7 shows the effect of the excitation frequency Ω represented in the calculations by the frequency ratio β for a two-span beam, where the velocity is held constant $(v = 15 \cdot 5$ m/s$)$. It is clear from Fig. 7 that the circular frequency Ω has visible effect on the shape of the dynamic displacement. Higher excitation frequency leads to sharper shapes and larger maximum deflections until a value of β . By increasing β until a certain value, the displacements increase. However, after that certain value of β , the displacements decrease with the increase in β . This situation is observed from Fig. 7 Additionally, the dynamic displacements for two-span beam are very small with respect to the single-span beam.

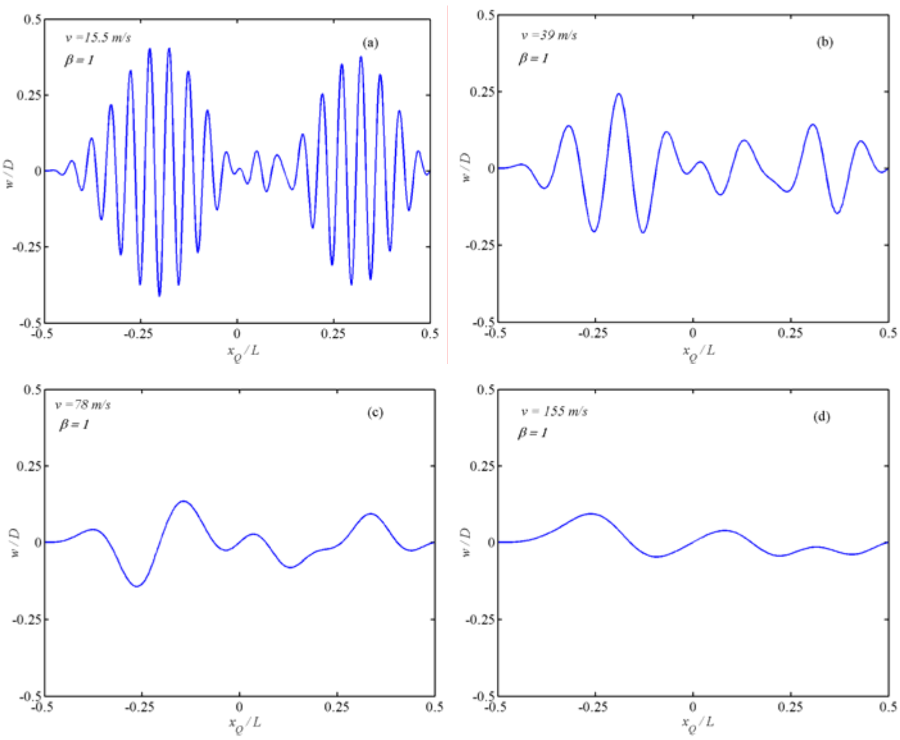

Figure 6. Normalized deflections under moving harmonic load for two-span beam for $v = 15 \cdot 5$, $v = 39$, $v = 78$, $v = 155$ m/s, $\xi = 0 \cdot 05$, $\beta = 1$

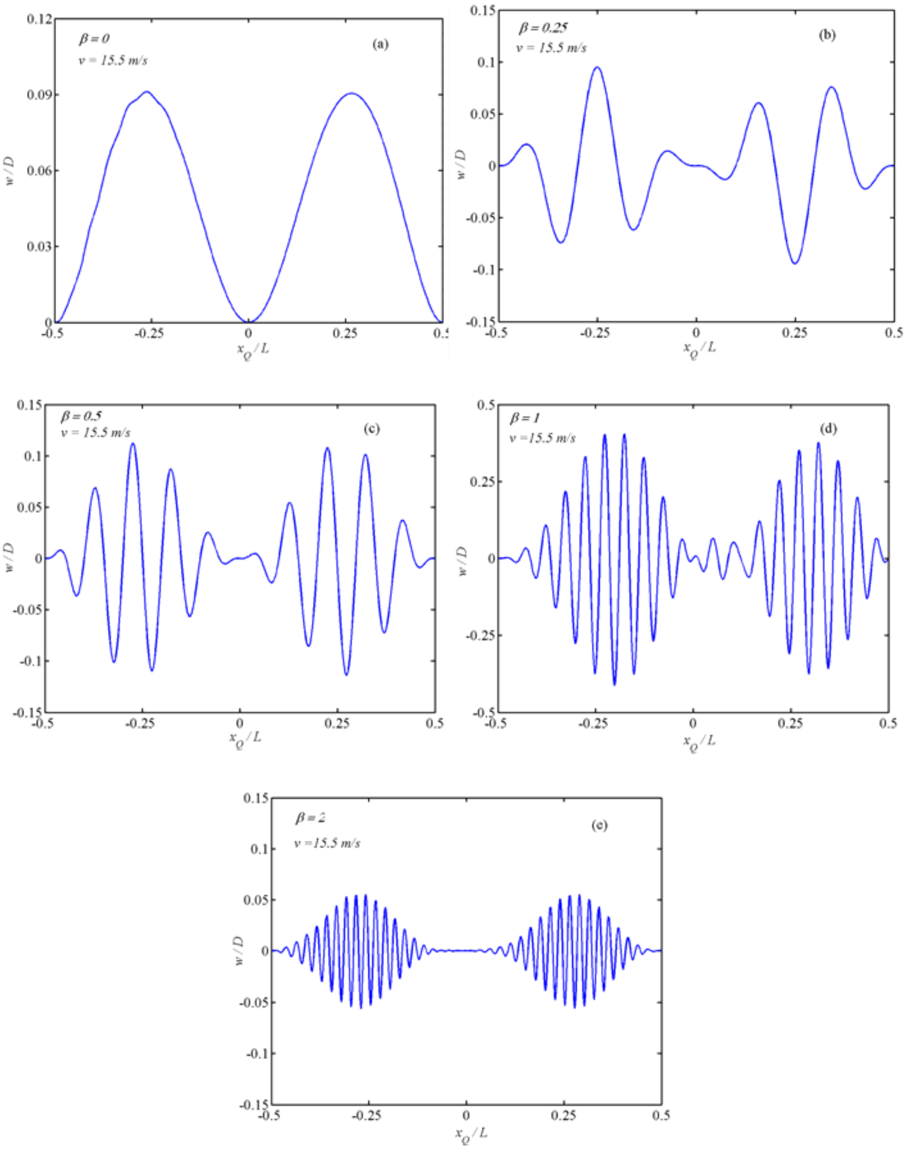

Figure 7. Normalized deflections under the moving harmonic load for two-span beam for $\beta = 0\cdot 0, 0\cdot 25, 0\cdot 50, 1\cdot 0, 2\cdot 0$, $\xi = 0\cdot 05$, $v = 15\cdot 5$ m/s

4. CONCLUSIONS

The dynamic deflections of beams subjected to a moving harmonic load with a constant velocity have been investigated. To use the Lagrange equations with the trial function in the polynomial form and to satisfy the constraint conditions by the use of Lagrange multipliers is a very good way for studying the dynamic behavior of continuous beams subjected to a moving harmonic load. Numerical calculations have been conducted to clarify the effects of the three important parameters, the axial velocity of the moving harmonic load, the excitation frequency of the moving load and the damping of the viscoelastic beam. It is observed from the investigations that the axial velocity of the load, the frequency of the load, the damping of the viscoelastic beam have a very important effect on the deflections.

All of the obtained results are very accurate and may be useful for designing structural and mechanical systems under moving harmonic loads.

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