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FRACTIONAL SUPERSYMMETRIC-*sl(2)*

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KESİRSEL SÜPERSİMETRİK- *sl(2)*

ÖZET

Permütasyon grubunun S_3 invaryant formları üzerinde kurulan Lie cebrinin kübik kökü Hopf cebri formalizminde ifade edildi. n=3 'te *sl(2)* 'nin N=4 kesirsel süper genellemesini gözönüne aldık.
Anahtar Sözcükler: Kesirsel süpercebirler, sl(2) Lie cebiri, Kesirsel süper- sl(2)

ABSTRACT

The 3rd root of Lie algebra based on the permütation group S_3 invariant forms is formulated in the Hopf algebra formalism. We consider N=4 fractional super generalizations of *sl(2)* at n=3
Keywords: Fractional superalgebras, sl(2) Lie algebra, Fractional super- sl(2)

1. INTRODUCTION

To arrive at a superalgebra one adds new elements Q_a to generators X_j of the corresponding Lie algebra and defines the relations

$$\{Q_a, Q_b\} = b_{ab}^j X_j \tag{1}$$

observing that the anticommutator in the above relation is invariant under the cyclic Z_2 or permütation S_2 groups anticommutator. To arrive at cubic root of a Lie algebra g, instead of (1) has the cubic relation

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$$Q_a Q_b Q_g + Q_g Q_a Q_b + Q_b Q_g Q_a = b^j_{abg} X_j \tag{2}$$

which is Z_3 invariant and the cubic relation

$$Q_a \{Q_b, Q_g\} + Q_b \{Q_a, Q_g\} + Q_g \{Q_a Q_b\} = b^j_{abg} X_j \tag{3}$$

which is S_3 invariant. From the above relations only (3) appears to be consistent at the co-algebra level. So we used the relation (3).

Fractional superalgebras based on S_n invariant form were first introduced in [1,2] and later constructed in the Hopf algebra context and defined their dual in [3]. In this paper, according to [3] we discuss fractional super- $sl(2)$ for N= 4.

There are other approaches to fractional supersymmetry in the Literature [4-9]. For example, one can arrive at fractional super algebras by using quantum groups at the roots of unity [10]. The plan of the paper is as follows. In the section 2, we give a formulation of fractional superalgebras in the Hopf algebra formalism from the [3]. In the section 3, we consider N=4 fractional supergeneralization of $sl(2)$ at n=3. we denoted this algebra by $U^4_3(sl(2))$.

2. REVIEW OF FRACTIONAL SUPERALGEBRAS

Let $U(g)$ be the universal enveloping algebra of a Lie algebra g generated by X_j j=1,2,..., dim(g) with

$$[X_i, X_j] = \sum_{k=1}^{\dim(g)} c^k_{ij} X_k \tag{4}$$

Where c^k_{ij} are the structure constants of the Lie algebra g . The Hopf algebra structure of $U(g)$ is given by

$$\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j, \qquad e(X_j) = 0, \qquad S(X_j) = -X_j. \tag{5}$$

To arrive at cubic root of $U(g)$. we shall use S_3 invariant form. Therefore, we defined an algebra generated by X_j , j=1,..., dim(g) and $Q_a, K, \ a = 1,\dots,N$ satisfying the relations (4) and

$$\{Q_a, Q_b, Q_g\} = b^j_{abg} X_j \tag{6}$$

$$[Q_a, X_j] = a^j_{ab} Q_b \tag{7}$$

and

$$K Q_a = q Q_a K \ , \ q^3 = 1 \ , \ K^3 = 1 \tag{8}$$

where

$$\{Q_a, Q_b, Q_g\} \equiv Q_a\{Q_b, Q_g\} + Q_b\{Q_a, Q_g\} + Q_g\{Q_a, Q_b\}$$

is the S_3 invariant form, c_{ij}^k and a_{ab}^j, b_{abg}^j are the structure coefficients satisfying the Jacobi and super Jacobi identities. This algebra is denoted by the symbol $U_3^N(g)$. The above algebra is a Hopf algebra with the following co structures [3]:

$$\Delta(Q_a) = Q_a \otimes 1 + K \otimes Q_a \quad , \quad \Delta(K) = K \otimes K \quad , \tag{9}$$

$$e(Q_j) = 0 \quad , \quad e(K) = 1 \quad , \tag{10}$$

$$S(Q_j) = -K^2 Q_j \quad , \quad S(K) = K^2 \ . \tag{11}$$

To define structure constant a_{ab}^j and b_{abg}^j we have to derive identities involving the commutator and S_3 invariant form. One can check that relations

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0 \tag{12}$$

$$[A,\{B,C,D\}] + \{[B,A],C,D\} + \{B,[C,A],D\} + \{B,C,[D,A]\} = 0 \tag{13}$$

and

$$[A,\{B,C,D\}] + [B,\{A,C,D\}] + [C,\{B,A,D\}] + [D,\{B,C,A\}] = 0 \tag{14}$$

are identically satisfed [3]. The relation (12) is the usual Jacobi identity.
Inserting

$$A = X_i \quad , \quad B = X_j \quad , \quad C = Q_a \tag{15}$$

into (12) and using (7) and (4) we get

$$\sum_{s=1}^{N} \left(a_{as}^i a_{sb}^j - a_{as}^j a_{sb}^i \right) = \sum_{k=1}^{\dim(g)} c_{ij}^k a_{ab}^k \tag{16}$$

Comparing the above relation with (4) we conclude that the $N \times N$ matrices $a^j \equiv \left(a_{ab}^j\right)_{a,b=1}^N$ define a N-dimensional representation of a given Lie algebra. Of course, these matrices are not unique.

Let us now consider restrictions on structure coefficients coming from the other identities. Inserting

$$A = X_k \quad , \quad B = Q_a \quad , \quad C = Q_b \quad , \quad D = Q_g \tag{17}$$

into the identity (13) we get

$$\sum_{s=1}^{N} \left(a_{as}^k b_{sbg}^i + a_{bs}^k b_{sag}^i + a_{gs}^k b_{sba}^i \right) = \sum_{j=1}^{\dim(g)} c_{jk}^i b_{abg}^j \tag{18}$$

and

$$A = Q_s \quad , \quad B = Q_a \quad , \quad C = Q_b \quad , \quad D = Q_g \tag{19}$$

into (14) and using (6), (7) we obtained following the relation

$$\sum_{k=1}^{\dim(g)} \left(b^k_{abg}\, a^k_{st} + b^k_{sab}\, a^k_{gt} + b^k_{gsa}\, a^k_{bt} + b^k_{bgs}\, a^k_{at} \right) = 0 \ . \tag{20}$$

3. N=4 FRACTIONAL SUPER-$sl(2)$

We know that the generators of the algebra $sl(2)$ satisfy the following commutation relations

$$[X_1, X_2] = X_3 \qquad [X_3, X_1] = 2X_1 \qquad [X_3, X_2] = -2X_2 \tag{21}$$

From the relation (21) one has

$$c^3_{12} = 1 \qquad c^1_{31} = 2 \qquad c^2_{32} = -2 \tag{22}$$

For N=4, the matrix $a^j = \{a^j_{ab}\}$ due to (16) is an arbitrary 4-dimensional representation of $sl(2)$. The solution of (18) and (20) for b^j_{abg} is fully determined by this representation. where b^j_{abg} is symmetric in a, b *and* g. We consider N=4 super generalization of sl(2) at n=3, that is $q = e^{i\frac{p}{3}}$. We have different superalgebras depending on the choice of a^j.

(i) we take $a^j_{ab} = 0$. Then the relations (18) and (20) imply $b^j_{abg} = 0$. The obtained structure constants imply that the fractional superalgebra $U^4_3(sl(2))$ is the direct product of the universal enveloping algebra $U(sl(2))$ with the Hopf algebra generated by Q_1, Q_2, Q_3, Q_4 and K satisfying the relations

$$K\,Q_a = q\,Q_a K \qquad \{Q_a, Q_b, Q_g\} = 0 \qquad K^3 = 1 \tag{23}$$

and the Hopf algebra structure (9)- (11).

(ii) Take the vector representation

$$a^1 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad a^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad a^3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{24}$$

The substitution of

$$a^1_{12} = a^2_{21} = a^1_{34} = a^2_{43} = \sqrt{3} \qquad a^1_{23} = a^2_{32} = 2 \tag{25}$$
$$a^3_{11} = 3, \quad a^3_{22} = 1, \quad a^3_{33} = -1, \qquad a^3_{44} = -3$$

into (20) and (18), will give all parameters b^{j}_{abg} are zero. Thus we obtained the fallowing fractional superalgebra :

$$\{Q_a , Q_b , Q_g \} = 0 \tag{26}$$

and

$$\begin{aligned}
&[Q_1, X_1] = \sqrt{3}Q_2 \qquad [Q_1, X_3] = 3Q_1 \\
&[Q_2, X_1] = 2Q_3 \qquad [Q_2, X_2] = \sqrt{3}Q_1 \qquad [Q_2, X_3] = Q_2 \\
&[Q_3, X_1] = \sqrt{3}Q_4 \qquad [Q_3, X_2] = 2Q_2 \qquad [Q_3, X_3] = -Q_3 \\
&[Q_4, X_2] = \sqrt{3}Q_3 \qquad [Q_4, X_3] = -3Q_1
\end{aligned} \tag{27}$$

Note that, for N=3 the relations (26) are not zero [3].

(iii) Assume that two of the fractional super generators Q_1, Q_2, Q_3 and Q_4 transform as spinors and the remaining two transforms as scalars, that is

$$a^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad a^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad a^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

The substitution of

$$a^1_{12} = a^2_{21} = a^3_{11} = 1 \qquad a^3_{22} = -1 \tag{29}$$

into (20) gives

$$\begin{aligned}
&b^1_{222} = -3b^2_{112} = 3b^3_{122} \\
&b^1_{122} = -\frac{1}{3}b^2_{111} = b^3_{112} \\
&b^1_{223} = -b^2_{113} = 2b^3_{123} \\
&b^1_{224} = -b^2_{114} = 2b^3_{124}
\end{aligned} \tag{30}$$

The substituting these into (18) one finds that the only solution is $b^{j}_{abg} = 0$. In this case, we obtained the following fractional superalgebra :

$$\{Q_a Q_b Q_g \} = 0 \ , \tag{31}$$

$$[Q_1, X_1] = Q_2 \ , \ [Q_2, X_2] = Q_1 \ , \ [Q_1, X_3] = Q_1 \ , \quad [Q_2, X_3] = -Q_2 \ . \tag{32}$$

Note that, for N=3 the relations (31) are not zero [3].

4. CONCLUSION

By applying fractional super algebra methods which are explained at Ref [3], to *sl(2)* Lie algebra which is a special case, we obtained N=3 fractional super generalization of *sl(2)* at n=3. In this generalization some $b^{j}_{\alpha\beta\gamma}$ structure constants where different from zero.

In this paper, by applying the same method we obtained N=4 fractional super generalization of *sl(2)* at n=3 and found $b^{j}_{\alpha\beta\gamma}$ structure constants equal to zero.

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