

## ON THE REGULARIZED TRACE OF THE DIFFERENTIAL OPERATOR EQUATION GIVEN IN A FINITE INTERVAL

Ehliman ADIGÜZELOV, Özlem BAKŞI\*

*Yıldız Teknik Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, Davutpaşa-İSTANBUL*

Geliş /Received: 17.06.2003 Kabul/Accepted: 07.01.2004

SONLU BİR ARALIKTA VERİLMİŞ DİFERANSİYEL OPERATÖR DENKLEMİNİN DÜZENLİ İZİ ÜZERİNE

### ÖZET

Bu çalışmada, sonlu bir aralıkta verilmiş sınırsız operatör katsayılı ikinci mertebeden diferansiyel operatörün düzenli izi için bir formül bulunmuştur.

**Anahtar Sözcükler:** Hilbert uzayı, Kendine eş operatör, çekirdek operatör

### ABSTRACT

In this work; a formula for the regularized trace of second order differential operator, which is given in a finite interval and with unbounded operator coefficient, is found.

**Keywords:** Hilbert space, Self-adjoints operator, Kernel operator

### 1. INTRODUCTION

Let  $H$  be a separable Hilbert space. We denote the inner product in  $H$  by  $(\cdot, \cdot)_H$  and the norm in  $H$  by  $\|\cdot\|_H$ . The function  $f$  is strongly measurable belonging to  $H$  defined on  $[0, p]$  and satisfies the

condition  $\int_0^p \|f(x)\|_H^2 dx < \infty$ . The set of all functions  $f$  is denoted by

$H_1 = L_2(H; [0, p])$ . If the inner product of arbitrary two elements  $f$  and  $g$  of the space  $H_1$  is defined as

$$(f, g) = \int_0^p (f(x), g(x))_H dx \quad (1.1)$$

then  $H_1$  becomes a separable Hilbert space, [1]. The norm in the space  $H_1$  is denoted by  $\|\cdot\|$ .  $S_\infty(H)$  denotes the set of compact operators from  $H$  to  $H$ . If  $B \in S_\infty(H)$  then  $B^*B$  is a non-negative self-adjoint operator and  $(B^*B)^{1/2} \in S_\infty(H)$ , [2]. Let the non-zero eigenvalues of the

\* Sorumlu Yazar/Corresponding Author: e-mail: [baksi@yildiz.edu.tr](mailto:baksi@yildiz.edu.tr) ; tel: (0212) 449 1528

operator  $(B^*B)^{1/2}$  be  $s_1 \geq s_2 \geq \dots \geq s_k$  ( $0 \leq k \leq \infty$ ). Here, each eigenvalue is repeated according to multiplicity. The numbers of  $s_1, s_2, \dots, s_k$  are called s-numbers of the operator B. If  $k < \infty$  then  $s_j = 0$  ( $j = k + 1, k + 2, \dots$ ) will be accepted. s-numbers of the operator B is also denoted by  $s_k(B)$  ( $k=1, 2, \dots$ ). If B is a normal operator, that is  $B^*B=BB^*$  then  $s_k(B) = |I_k(B)|$  ( $k=1, 2, \dots$ ), [2].

Here,  $I_1(B), I_2(B), \dots, I_k(B)$  are the non-zero eigenvalues of the operator B.  $S_1$  or  $S_1(H)$  is the set of all the operators  $B \in S_\infty(H)$  the s-numbers of which satisfy the condition

$\sum_{k=1}^\infty s_k(B) < \infty$ . The set  $S_1$  is a separable Banach space [2] with respect to the norm

$$\|B\|_{S_1(H)} = \sum_{k=1}^\infty s_k(B) \quad B \in S_1 \tag{1.2}$$

An operator is called a kernel operator if it belongs to  $S_1(H)$ . If the operator  $A \in S_1(H)$  and the operator  $B : H \rightarrow H$  is linear and bounded then  $AB, BA \in S_1(H)$  and

$$\|BA\|_{S_1(H)} \leq \|B\| \|A\|_{S_1(H)}, \quad \|AB\|_{S_1(H)} \leq \|B\| \|A\|_{S_1(H)} \tag{1.3}$$

[2]. If  $B \in S_1(H)$  and  $\{e_k\}_{k=1}^\infty \subset H$  is any orthonormal basis, the series  $\sum_{k=1}^\infty (Be_k, e_k)_H$  is convergent

and the sum of the series  $\sum_{k=1}^\infty (Be_k, e_k)_H$  does not depend on the choice of the basis  $\{e_k\}_{k=1}^\infty$ . The

sum of the series  $\sum_{k=1}^\infty (Be_k, e_k)_H$  is said to be matrix trace and is denoted by  $trB$ . If  $A, B \in S_1(H)$

and  $a, b$  is any scalar then

$$tr(aA + bB) = atrA + btrB, \quad trA^* = \overline{trA}, \quad \text{and} \quad trB = \sum_{k=1}^{u(A)} I_k(B) \tag{1.4}$$

In the last equality, each eigenvalue is added according to its own algebraic multiplicity number.  $u(A)$  denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator B,

[2]. The sum of the series  $\sum_{k=1}^{u(A)} I_k(B)$  is said to be the spectral trace of the operator B. Recall that a

self-adjoint operator is said to have purely-discrete spectrum if its spectrum consist of eigenvalues  $\{I_j\}_{j=1}^\infty$  of finite multiplicity and  $\lim_{n \rightarrow \infty} |I_n| = \infty$ .

Let us consider the differential expression in the space  $H_1 = L_2(H, [0, p])$

$$l_0(y) = -y''(x) + Ay(x) \tag{1.5}$$

Here, a densely defined operator  $A : D(A) \rightarrow H$  in the space H satisfies the conditions

$$A = A^* \geq I, \quad A^{-1} \in S_\infty(H), \quad (I \text{ is identity operator in } H)$$

Let  $g_1 \leq g_2 \leq \dots \leq g_j \leq \dots$  be the eigenvalues of the operator A and  $j_1, j_2, \dots, j_j, \dots$  be the orthonormal eigenvectors corresponding to these eigenvalues.

Moreover,  $D_0$  denotes the set of the functions  $y(x)$  satisfying the conditions:

(1)  $y(x)$  has continuous derivative of the second order with respect to the norm in the space H in the interval  $[0, p]$

(2)  $Ay(x)$  is continuous with respect to the norm in the space H.

**On the Regularized Trace of the Differential...**

(3)  $y'(0) = y'(p) = 0$

Here  $\overline{D_0} = H_1$  ( $\overline{D_0}$  denoted by closure of  $D_0$ ) and the operator  $L_0' = D_0 \rightarrow H_1$ ,  $L_0' y = l_0(y)$  is symmetric. The eigenvalues of  $L_0'$  are  $k^2 + g_j$  ( $k=0,1,2,\dots; j=1,2,\dots$ ) and the orthonormal eigenvectors corresponding to these eigenvalues are  $M_k \cos kx_j$  ( $k=0,1,2,\dots; j=1,2,\dots$ ). Here,

$$M_k = \begin{cases} \sqrt{p^{-1}} & ; \text{if } k = 0 \\ \sqrt{2p^{-1}} & ; \text{if } k = 1,2,\dots \end{cases} \tag{1.6}$$

As seen, the orthonormal eigenvectors system of the symmetric operator  $L_0'$  is an orthonormal basis in the space  $H_1$ .

Let  $Q(x)$  be an operator function satisfying the following conditions:

(1)  $Q(x)$  has weak derivative of the second order in the interval  $[0, p]$ . The operator function  $Q''(x)$  is weakly measurable, and for every  $x \in [0, p]$ ,  $Q^{(i)}(x) : H \rightarrow H$  ( $i = 0,1,2$ ) are self-adjoint compact operators.

(2) The functions  $\|Q^{(i)}(x)\|_{S_1(H)}$  ( $i = 0,1,2$ ) are bounded and measurable in the interval  $[0, p]$ .

(3) For every  $f \in H$   $\int_0^p (Q(x)f, f)_H dx = 0$ .

In this work, we find a formula for the eigenvalues of the operators  $L_0 = \overline{L_0'}$  and  $L = L_0 + Q$  and this formula is said to be regularized trace formula.

The regularized trace formulas for scalar differential operators are studied in [3],[4],[5] and in many other works. The list of the works on the subjects is given in [6] and [7], but a small number of these works are on the regularized trace of differential operators with operator coefficient.

In [8], the regularized trace of the Sturm-Liouville operator with bounded operator coefficient is calculated. In [9], a formula for the regularized trace of the difference of two Sturm-Liouville operators which is given in half-axis with the bounded operator coefficient is found. In [10], a formula for the regularized trace of the Sturm-Liouville operator under Dirichlet boundary conditions with unbounded operator coefficient, is found. In [11], the regularized trace of a singular differential operator of second order with bounded operator coefficient is investigated. In [12] and [13], the formulas for the regularized traces of differential operators with bounded operator coefficient are found.

**2. SOME RELATIONS ABOUT THE EIGENVALUES AND RESOLVENTS**

In this section, we will prove that the operators  $L_0$  and  $L$  are self-adjoint and we will find some relations about the eigenvalues and resolvents of the operators  $L_0$  and  $L$ .

**Theorem 2.1.** Every symmetric closed operator, the eigenvectors system of which is closed is self-adjoint.

**Proof .** Let  $H$  be a separable Hilbert space. Let  $B : D(B) \rightarrow H$  be a symmetric operator with  $D(B) \subset H$ ,  $\{e_i\}_1^\infty$  be an orthonormal system consisting of the eigenvectors of the operator  $B$

and  $I$  also be an nonreal number. Since  $(B - II)^{-1}$  is a bounded closed operator, the linear manifold  $D((B - I I)^{-1}) = R(B - I I)$  is closed. That is, the linear manifold  $R(B - I I)$  is a subspace of  $H$ . On the other hand, since the subspace  $R(B - I I)$  contains the closed system  $\{e_i\}_1^\infty$ , then  $R(B - I I) = H$  similarly,  $R(B - \bar{I}I) = H$

In this case, as well known, the operator  $B$  is self-adjoint. The Theorem 2.1 is proved. •

Since the eigenvectors system of the symmetric operator  $L_0$  is closed, according to the Theorem 2.1. the operator  $L_0 = \overline{L_0}$  is self-adjoint and since the bounded operator  $Q : H_1 \rightarrow H_1$  is self-adjoint, the operator  $L = L_0 + Q$  is also self-adjoint.

The operators  $L_0$  and  $L$  have purely-discrete spectrum. Let the eigenvalues of the operators  $L_0$  and  $L$  be  $m_1 \leq m_2 \leq \dots \leq m_n \leq \dots$  and  $I_1 \leq I_2 \leq \dots \leq I_n \leq \dots$  respectively.

By using [14], we can prove the following theorem:

**Theorem 2.2.** If  $g_j \sim aj^a$  as  $j \rightarrow \infty$  that is

$$\lim_{j \rightarrow \infty} \frac{g_j}{aj^a} = 1, \text{ then as } n \rightarrow \infty \quad I_n, m_n \sim dn^{\frac{2a}{2+a}} \quad (d > 0)$$

By using Theorem 2.2., it is easily seen that the sequence  $\{m_n\}$  has a subsequence  $m_{n_1} < m_{n_2} < \dots < m_{n_m} < \dots$  such that

$$m_k - m_{n_m} \geq d_0 \left( k^{\frac{2a}{2+a}} - n_m^{\frac{2a}{2+a}} \right) \quad (k = n_m, n_m + 1, n_m + 2, \dots)$$

Let  $R_l^0 = (L_0 - II)^{-1}$  and  $R_l = (L - II)^{-1}$  be the resolvents of the operators  $L_0$  and  $L$

respectively. If  $a > 2$  by Theorem 2.2,  $R_l^0$  and  $R_l$  are compact operators for  $l \neq m_n, I_n$  ( $n=1,2,3,\dots$ ). In this case

$$tr(R_l - R_l^0) = trR_l - trR_l^0 = \sum_{k=1}^{\infty} \left( \frac{1}{I_k - l} - \frac{1}{m_k - l} \right) \quad (2.1)$$

[2]. Let  $b_m = 2^{-1}(m_{n_m} + m_{n_m+1})$ . It easy to see that for the large value of  $m$  the inequalities

$$m_{n_m} < b_m < m_{n_m+1}, \quad I_{n_m} < b_m < I_{n_m+1} \quad \text{are satisfied and the series}$$

$$\sum_{k=1}^{\infty} \left( \frac{1}{I_k - l} \right), \quad \sum_{k=1}^{\infty} \left( \frac{1}{m_k - l} \right)$$

are uniform convergent on the circle  $|l| = b_m$ . Therefore by (2.1)

$$\sum_{k=1}^{n_m} (I_k - m_k) = -\frac{1}{2\pi i} \int_{|l|=b_m} l tr(R_l - R_l^0) dl \quad (2.2)$$

**Lemma 2.1.** If  $g_j \sim a \cdot j^a$  ( $0 < a < \infty, 2 < a < \infty$ ) as  $j \rightarrow \infty$  then

$$\|R_l^0\|_{S_1(H_1)} \leq const \cdot n_m^{1-d} \quad \left( d = \frac{a-2}{a+2} \right) \text{ on the circle } |l| = b_m.$$

**On the Regularized Trace of the Differential...**

**Proof .** For  $I \notin \{m_k\}_{k=1}^{\infty}$  since  $R_I^0$  is a normal operator then

$$\|R_I^0\|_{S_1(H_1)} = \sum_{k=1}^{\infty} \frac{1}{|m_k - I|}$$

[2]. Since  $|I| = b_m = 2^{-1}(m_{n_m} + m_{n_m+1})$  then

$$\begin{aligned} \|R_I^0\|_{S_1(H_1)} &\leq \sum_{k=1}^{\infty} \frac{1}{|I| - m_k} \leq \sum_{k=1}^{n_m} \frac{2}{m_{n_m} + m_{n_m+1} - 2m_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{2m_k - m_{n_m} - m_{n_m+1}} \\ &\leq \sum_{k=1}^{n_m} \frac{2}{m_{n_m+1} - m_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{m_k - m_{n_m}} \end{aligned} \quad (2.3)$$

is obtained. By using the Theorem 2.2, we limit the sums on the right hand side of the inequality above:

$$\sum_{k=1}^{n_m} \frac{1}{m_{n_m+1} - m_k} < \frac{n_m}{m_{n_m+1} - m_{n_m}} \leq \frac{n_m}{d_0[(n_m + 1)^{1+d} - n_m^{1+d}]} < \frac{n_m}{d_0 n_m^d} = d_0^{-1} n_m^{1-d} \quad (2.4)$$

$$\begin{aligned} \sum_{k=n_m+1}^{\infty} \frac{1}{m_k - m_{n_m}} &\leq d_0^{-1} \sum_{k=n_m+1}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}} \\ &= \frac{1}{d_0[(n_m + 1)^{1+d} - n_m^{1+d}]} + d_0^{-1} \sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}} \end{aligned} \quad (2.5)$$

Moreover

$$\sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+d} - n_m^{1+d}} \leq \int_{n_m+1}^{\infty} \frac{dx}{x^{1+d} - n_m^{1+d}}$$

and it is easily shown that  $\int_{n_m+1}^{\infty} \frac{dx}{x^{1+d} - n_m^{1+d}} \leq d^{-1} n_m^{-\frac{d^2}{1+d}}$ .

Considering the last two inequalities in (2.5)

$$\sum_{k=n_m+1}^{\infty} \frac{1}{m_k - m_{n_m}} \leq \frac{1}{d_0[(n_m + 1)^{1+d} - n_m^{1+d}]} + \frac{n_m^{-\frac{d^2}{1+d}}}{d_0 d} \leq \frac{2}{d_0 d} \quad (2.6)$$

By (2.3), (2.4) and (2.6)

$$\|R_I^0\|_{S_1(H_1)} \leq \frac{6}{d_0 d} \cdot n_m^{1-d} \text{ is found. Lemma 2.1 is proved } \bullet .$$

**Lemma 2.2.** If  $g_j \sim a \cdot j^a$  ( $0 < a < \infty$ ,  $2 < a < \infty$ ) as  $j \rightarrow \infty$  and Q is a bounded self adjoint operator from  $H_1$  to  $H_1$  then,  $|I| = b_m$  and for the large values of m  $\|R_I\| \leq \text{const} \cdot n_m^{-d}$

**Proof .** Since the eigenvalues of the kernel operator  $R_I$  are  $\{(I_k - I)^{-1}\}_{k=1}^{\infty}$  then

$$\|R_I\| = \max_k \{ |I_k - I|^{-1} \} \quad (2.7)$$

For  $|I| = b_m$

$$\| |I_k| - |I| \| = \left| |I_k| - \frac{1}{2} \cdot (m_{n_m} + m_{n_m+1}) \right| = \frac{1}{2} \| m_{n_m} + m_{n_m+1} - 2|I_k| \| \tag{2.8}$$

$$m_{n_m} + m_{n_m+1} - 2|I_k| \geq m_{n_m} + m_{n_m+1} - 2I_{n_m} = m_{n_m+1} - m_{n_m} + 2(m_{n_m} - I_{n_m})$$

$k \leq n_m$  and for the large values of  $m$ , since  $|I_k| < I_{n_m}$  then

$$m_{n_m} + m_{n_m+1} - 2|I_k| \geq m_{n_m+1} - m_{n_m} - 2|m_{n_m} - I_{n_m}| > m_{n_m+1} - m_{n_m} - c \tag{2.9}$$

Here  $c$  is a constant.  $k \geq n_m + 1$  and for the large values of  $m$ , since  $|I_k| = I_k \geq I_{n_m+1}$  then

$$\begin{aligned} 2|I_k| - m_{n_m} - m_{n_m+1} &\geq 2I_{n_m+1} - m_{n_m} - m_{n_m+1} \\ &= 2(I_{n_m+1} - m_{n_m+1}) + m_{n_m+1} - m_{n_m} \\ &\geq m_{n_m+1} - m_{n_m} - 2|I_{n_m+1} - m_{n_m+1}| \geq m_{n_m+1} - m_{n_m} - c \end{aligned} \tag{2.10}$$

On the other hand, since  $\lim_{m \rightarrow \infty} (m_{n_m+1} - m_{n_m}) = \infty$  by (2.8), (2.9) and (2.10)

$$\| |I_k| - |I| \| \geq \frac{1}{4} (m_{n_m+1} - m_{n_m}) \text{ is found. By using the Theorem 2.2}$$

$$\| |I_k| - |I| \| \geq \frac{d_0}{4} [(n_m + 1)^{1+d} - n_m^{1+d}] > \frac{d_0}{4} n_m^d \text{ or } \| |I_k| - |I| \|^{-1} < \frac{4}{d_0} n_m^{-d}$$

is obtained. By (2.7) and the last inequality  $\|R_I\| \leq \frac{4}{d_0} \cdot n_m^{-d}$  is found. Lemma is proved. •

We will use the last two lemmas to prove the following theorem.

### 3. THE FORMULA FOR THE REGULARIZED TRACE OF THE OPERATOR L

This is a well known formula for the resolvents of the operators  $L_0$  and  $L$  :

$$R_I = R_I^0 - R_I Q R_I^0 \quad (I \in r(L) \cap r(L_0)) .$$

By using the last formula and (2.2), it can be shown that

$$\sum_{k=1}^{n_m} (I_k - m_k) = \sum_{j=1}^p D_{mj} + D_m^{(p)} \tag{3.1}$$

Here

$$D_{mj} = \frac{(-1)^j}{2^j p^j} \int_{|I|=b_m} \text{tr} [(Q R_I^0)^j] dI \tag{3.2}$$

$$D_m^{(p)} = \frac{(-1)^p}{2^p p^p} \int_{|I|=b_m} I \cdot \text{tr} [R_I (Q R_I^0)^{p+1}] dI \tag{3.3}$$

**Theorem 3.1.** If  $g_j \sim a \cdot j^a$  ( $0 < a < \infty$ ,  $2 < a < \infty$ ) as  $j \rightarrow \infty$ , and the operator function  $Q(x)$  satisfies the conditions (1) and (2) then

$$\lim_{m \rightarrow \infty} D_{mj} = 0 \quad (j \geq 2)$$

**Proof.** According to the formula (3.2)

$$|D_{mj}| \leq \frac{1}{2^j p^j} \int_{|I|=b_m} |\text{tr} (Q R_I^0)^j| dI \leq \frac{1}{2^j p^j} \int_{|I|=b_m} \|Q R_I^0\|_{S_1(H_1)} \|Q R_I^0\|^{j-1} dI$$

**On the Regularized Trace of the Differential...**

$$\begin{aligned}
 |D_{mj}| &\leq \frac{1}{2pj} \int_{|l|=b_m} \|Q\| \cdot \|R_l^0\|_{S_1(H_1)} \|QR_l^0\|^{j-1} |dI| \\
 &\leq \frac{1}{2pj} \int_{|l|=b_m} \|Q\|^j \cdot \|R_l^0\|_{S_1(H_1)} \|R_l^0\|^{j-1} |dI| \leq \text{const} \int_{|l|=b_m} \|R_l^0\|_{S_1(H_1)} \|R_l^0\|^{j-1} |dI|
 \end{aligned} \tag{3.4}$$

Since  $R_l = R_l^0$  for  $Q \equiv 0$  according to Lemma 2.2

$$\|R_l^0\| \leq \text{const} \cdot n_m^{-d} \tag{3.5}$$

By using Lemma 2.1 and the inequalities (3.4), (3.5)

$$|D_{mj}| \leq \text{const} \int_{|l|=b_m} n_m^{1-d} \cdot n_m^{-d(j-1)} \cdot |dI| \leq \text{const} \cdot m_{n_m} \cdot n_m^{1-dj}$$

is obtained. Since  $m_{n_m} \leq \text{const} \cdot n_m^{1+d}$ , then  $|D_{mj}| \leq \text{const} \cdot n_m^{2-d(j-1)}$  is found. As seen; if

$j \geq \lceil 2d^{-1} \rceil + 2$  then

$$\lim_{m \rightarrow \infty} D_{mj} = 0 \tag{3.6}$$

It is necessary to show that the equality above is satisfied for  $j=2,3,\dots, \lceil 2d^{-1} \rceil + 1$  to complete the proof. Let us show for  $j=2$ .  $D_{m2}$  satisfies the following inequality

$$\begin{aligned}
 D_{m2} &= \sum_{k=1}^{n_m} \sum_{j=n_m+1}^{\infty} (m_k - m_j)^{-1} (j_k, Qj_j)_{H_1} (Qj_j, j_k)_{H_1}. \text{ Therefore} \\
 |D_{m2}| &\leq \sum_{j=n_m+1}^{\infty} \left[ (m_j - m_{n_m})^{-1} \sum_{k=1}^{\infty} \left| (Qj_j, j_k)_{H_1} \right|^2 \right]
 \end{aligned} \tag{3.7}$$

Since  $\{j_k\}_{k=1}^{\infty}$  is an orthonormal basis in the space  $H_1$  and the equality

$$\lim_{n \rightarrow \infty} \sum_{j=n_m+1}^{\infty} (m_j - m_{n_m})^{-1} = 0 \tag{3.8}$$

is satisfied, by (3.7) and (3.8)  $\lim_{n \rightarrow \infty} D_{m2} = 0$  is obtained. •

**Theorem 3.2.** If  $g_j \sim a \cdot j^a$  ( $0 < a < \infty$ ,  $2 < a < \infty$ ) as  $j \rightarrow \infty$  and  $Q(x)$  satisfies the conditions (1), (2), (3) then the formula in the form

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} (I_k - m_k) = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(p)]$$

is satisfied for the regularized trace of the operator L.

**Proof.** According to the formula (3.2)

$$D_{m1} = -\frac{1}{2pi} \int_{|l|=b_m} \text{tr}(QR_l^0) dI \tag{3.9}$$

Since the orthonormal eigenvectors system  $\{\Psi_p(x)\}_{p=1}^{\infty}$  according to the eigenvalues  $\{m_p\}_{p=1}^{\infty}$  of the operator  $L_0$  is an orthonormal basis of the space  $H_1$  then

$\text{tr}(QR_l^0) = \sum_{p=1}^{\infty} (QR_l^0 \Psi_p, \Psi_p)_{H_1}$ . If the equality above is written into the equality (3.9),

$$R_l^0 \Psi_p = (L_0 - lI)^{-1} \cdot \Psi_p = (m_p - l)^{-1} \Psi_p \text{ and } m_{n_m} < b_m < m_{n_m+1}$$

are considered, then  $D_{m1} = -\frac{1}{2\pi i} \int_{|l|=b_m} \left[ \sum_{p=1}^{\infty} (m_p - l)^{-1} (\mathcal{Q}\Psi_p, \Psi_p)_{H_1} \right] dl$

$$= \sum_{p=1}^{\infty} (\mathcal{Q}\Psi_p, \Psi_p)_{H_1} \cdot \frac{1}{2\pi i} \int_{|l|=b_m} \frac{dl}{l - m_p} = \sum_{p=1}^{n_m} (\mathcal{Q}\Psi_p, \Psi_p)_{H_1} \tag{3.10}$$

is obtained. Since the orthonormal eigenvectors according to the eigenvalues  $k^2 + g_j$  ( $k=0,1,2,\dots; j=1,2,\dots$ ) of the operator  $L_0$  are  $M_k \cos kxj_j$  ( $k=0,1,2,\dots; j=1,2,\dots$ ) respectively then  $\Psi_p(x) = M_{k_p} \cos k_p xj_{j_p}$  ( $p=1,2,\dots$ ) Therefore by (3.10)

$$D_{m1} = \sum_{p=1}^{n_m} \int_0^p (\mathcal{Q}(x)M_{k_p} \cos k_p xj_{j_p}, M_{k_p} \cos k_p xj_{j_p})_H dx$$

$$= \frac{1}{2} \sum_{p=1}^{n_m} M_{k_p}^2 \int_0^p (1 + \cos 2k_p x) (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H dx$$

is found.  $Q(x)$  satisfies the condition (3) and  $M_k = \sqrt{2p^{-1}}$  ( $k=1,2,\dots$ ) is considered, by the last relation

$$D_{m1} = \frac{1}{p} \sum_{p=1}^{n_m} \int_0^p \cos 2k_p x (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H dx \tag{3.11}$$

is found. If the operator function  $Q(x)$  satisfies the conditions (1) and (2), the multiple series

$\sum_{k=1}^{\infty} \sum_{j=1}^p (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H \cos 2kx dx$  is absolute convergent. In this case as known

$$\lim_{m \rightarrow \infty} \sum_{p=1}^{n_m} \int_0^p \cos 2k_p x (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H dx = \sum_{k=1}^{\infty} \sum_{j=1}^p \int_0^p (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H \cos 2kx dx$$

By (3.11) and the last equality

$$\lim_{m \rightarrow \infty} D_{m1} = \frac{1}{p} \sum_{k=1}^{\infty} \sum_{j=1}^p \int_0^p (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H \cos 2kx dx$$

$$= \frac{1}{4} \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \left[ \int_0^p (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H \cos kx dx \right] \cos k \cdot 0 \right.$$

$$\left. + \sum_{k=1}^{\infty} \left[ \int_0^p (\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H \cos kx dx \right] \cos kp \right\}$$

is found. If we consider that  $Q(x)$  satisfies the condition (3), the sums according to  $k$  on the right hand side of the last relation are the values at the point 0 and  $p$  of the Fourier series of the function

$(\mathcal{Q}(x)j_{j_p} \cdot j_{j_p})_H$  having second order derivative according to the functions  $\{\cos kx\}_{k=0}^{\infty}$  in the interval  $[0, p]$  respectively. Therefore

$$\lim_{m \rightarrow \infty} D_{m1} = \frac{1}{4} \sum_{j=1}^{\infty} [(\mathcal{Q}(0)j_{j_p} \cdot j_{j_p})_H + (\mathcal{Q}(p)j_{j_p} \cdot j_{j_p})_H]$$

or



$$\lim_{m \rightarrow \infty} D_{m1} = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(p)] \quad (3.12)$$

similar to the proof of the equality (3.6) the formula

$$\lim_{m \rightarrow \infty} D_m^{(p)} = 0 \quad \left( p \right) \frac{3(a+2)}{a-2} \quad (3.13)$$

can be proved. By the formulas (3.1) , (3.12) , (3.13) and Theorem 3.1

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} (I_k - m_k) = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(p)]$$

is obtained. The Theorem is proved.

## REFERENCES

- [1] Kirillov, A.A., *Elementary Theory Representations*, Springer verlag, New York, 1976.
- [2] Cohberg, C. and Krein, M.G., *Introduction to the Theory Linear non-self Adjoint Operators*, Translation of Mathematical Monographs, Vol.18 (AMS, Providence, R.I.,1969).
- [3] Gelfand, I.M.and Levitan, B.M. “ On a formula for eigenvalues of a differential operator of second order”, Dokl.Akad.Nauk SSSR, 1953, T.88, No:4, 593-596.
- [4] Dikiy, L.A., “About a formula of Gelfand-Levitan”, Usp.Mat.Nauk, 8(2), 119-123 (1953).
- [5] Halberg,C.J.and Kramer,V.A ”A generalization of the trace concept”, Duke Math.J.27(4),607-618 (1960).
- [6] Levitan,B.M.and Sargsyan,I.S.,*Sturm-Liouville and Dirac Op.*,Kluwer,Dordrecht, 1991.
- [7] Fulton, T.C. and Pruess, S.A., ”Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems”, J.Math.Anal.Appl.188, 297-340 (1994).
- [8] Chalilova, R.Z., ” On arranging Sturm-Liouville operator equation’s trace”, Funks.analiz, teoriya funktsii i ik pril.-Mahachkala, Vol.1, No:3, 1976.
- [9] Adıgüzelov, E.E.,” About the trace of the difference of two Sturm-Liouville operators with operator coefficient”, iz.An Az SSR, seriya fiz-tekn. i mat.nauk, No:5, 20-24, 1976.
- [10] Maksudov,F.G.,Bairamoglu,M. and Adıgüzelov, E.E. “On a regularized traces of the Sturm-Liouville operator on a finite interval with the unbounded operator coefficient “, Dokl.Akad, Nauk SSSR, English translation, Soviet Math, Dokl, 30(1984),No1,169-173.
- [11] Bairamoglu, M. and Adıgüzelov , E.E. “On a regularized trace formula for the Sturm-Liouville operator with a bounded operator coefficient and with a singularity” Differential Equations ,32(1996),no 12,1581-1585 (1997).
- [12] Adıgüzelov, E.E., Bayramov, A. and Baykal, O. “On the spectrum and regularized trace of the Sturm-Liouville problem with spectral parameter on the boundary condition and with the operator coefficient”, International Journal of differential Equations and Applications“ Vol.2,No3,2001, 317-333.
- [13] Adıgüzelov, E.E. Avcı, H. and Gul, E., ”The trace formula for Sturm-Liouville operator with operator coefficient”, J.Math.Phsy,Vol.42,1611-1624 No:6, 2001.
- [14] Gorbachuk, V.I. “About the asymptotic behaviour of the eigenvalues of boundary value problems for differential equations in the vector function space”, Ukr.Matem.Journal,T.27,No:5, 657-664, 1975.