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ON THE GENERALIZATION OF CARTESIAN PRODUCT OF FUZZY SUBGROUPS AND IDEALS

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BULANIK ALT GRUPLARIN VE İDEALLERİN KARTEZYEN ÇARPIMLARININ GENELLEŞTİRİLMESİ

ÖZET

Bu çalışmada Malik ve Mordeson'un makalesi genelleştirildi. Yani farklı grupların (halkaların) bulanık alt gruplarının (bulanık ideallerinin) kartezyen çarpımları incelendi. G_1 ve G_2 boştan farklı iki grup olmak üzere eğer m_1 ve m_2 G_1 ve G_2 (R_1 ve R_2 birimli olmak zorunda olmayan değişmeli iki halka olmak üzere) nin bulanık alt grupları(bulanık idealleri) ise kartezyen çarpımları $m_1 \times m_2$ da $G_1 \times G_2$ ($R_1 \times R_2$) nin bulanık alt grubudur (bulanık idealidir). Yukarıdaki ifadesinin ters yönleri de çalışılmıştır. Bu ifadeleri n farklı grup (halka) için de genelleştirilmiştir.
Anahtar Sözcükler: Bulanık alt küme, Bulanık Alt grup, Seviye alt grubu, Bulanık ideal, Seviye ideali, Bulanık bağıntı, Kartezyen çarpım

ABSTRACT

In this work I generalize Malik and Mordeson's paper [3]. I analysis the cartesian product of fuzzy subgroups (ideals) of two groups (two commutative rings Rings which have not necessarily identity element). That is; if m and s are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2) respectively then $m \times s$ is a fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). Conversely the opposite direction of the above statements is studied. We generalize the above statements for n different Groups (Rings).
Keywords: Fuzzy subset, Fuzzy subgroup, Level subgroup, Fuzzy ideal, Level ideal, Fuzzy relation, Cartesian product

1. INTRODUCTION

The concept of a fuzzy subset was introduced by Zadeh[5]. Fuzzy subgroup and its important properties were defined and established by Rosenfeld[2]. Then many authors have studied about it. After this time it was necessary to define fuzzy ideal of a ring. The notion of a fuzzy ideal of a ring was introduced by Liu [1]. Malik, Mordeson and Mukherjee have studied fuzzy ideals. The

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concept of a fuzzy relation on a set was introduced by Zadeh[6]. Bhattacharya and Mukherjee have studied fuzzy relation on groups. Malik and Mordeson [3] studied fuzzy relation on rings. Moreover Malik and Mordeson have written very important book for Fuzzy algebra which is "Fuzzy Commutative Algebra"[4].

In this paper G_i $(i=1,2,...,n)$ is a group and R_i $(i=1,2,...,n)$ is a commutative ring. A fuzzy relation on R is the fuzzy subset of $R \times R$. In our paper the cartesian product of two sets G_1 and G_2 (R_1 and R_2) is defined like that:
$\forall (a_1,b_1),(a_2,b_2) \in G_1 \times G_2\,(R_1 \times R_2)$ $(a_1,b_1)+(a_2,b_2)) = (a_1+a_2, b_1+b_2)$,
$(a_1,b_1).(a_2,b_2)) = (a_1a_2, b_1b_2)$. I generalize Malik and Mordeson's paper. That is; if m_1, m_2 are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2) respectively then $m_1 \times m_2$ is a fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). Let m_1, m_2 be fuzzy subsets of G_1, G_2 respectively such that $m_1 \times m_2$ is a fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). Then m_1 or m_2 is fuzzy subgroup (ideal) of G_1 or G_2 (R_1 or R_2) respectively. Let m_1 and m_2 be fuzzy subsets of R such that $m_1 \times m_2$ is a fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). If $\forall x \in G_1, \forall y \in G_2$ $m_1(e_1) = m_2(e_2)$, $m_1(x) \le m_1(e_1)$ and $m_2(y) \le m_2(e_2)$ ($\forall x \in R_1, \forall y \in R_2$ $m_1(0_1) = m_2(0_2)$, $m_1(x) \le m_1(0_1)$ and $m_2(y) \le m_2(0_2)$) then both m_1 and m_2 are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2). Also I extend these above theorems for n different Groups (Rings). That is if $m_1, m_2, m_3, ..., m_n$ are fuzzy subgroups (ideals) of $G_1, G_2, ..., G_n$ ($R_1, R_2, ..., R_n$) respectively , then $m_1 \times m_2 \times m_3 \times ... \times m_n$ is fuzzy subgroup (ideal) of $G_1 \times G_2 \times ... \times G_n$ ($R_1, R_2, ..., R_n$). Then I prove the opposite direction of the previous statement under some conditions.

2. PRELIMINARIES

In this section, we review some basic definitions and results.

Definition 2.1: *A* fuzzy subset of non empty set S is a function $m : S \to [0,1]$.

Definition 2.2: *A* fuzzy subset m of a group G is called a fuzzy subgroup of G if
(i) $m(xy) \ge \min(m(x), m(y))$
(ii) for all $x, y \in G$ $m(x^{-1}) \ge m(x)$.
If m is a fuzzy subgroup of G then $m(x^{-1}) = m(x)$ for all $x \in G$.

Definition 2.3: If m is a fuzzy subset of S , then for any $t \in \operatorname{Im} m$, the set $m_t = \{x \in S \mid m(x) \ge t\}$ is called the level subset of S with respect to m .

Theorem 2.4 ([1] **):** Let m be fuzzy subset of G . m is a fuzzy subgroup of G if and only if m_t is an subgroup of G for $\forall t \in \operatorname{Im} m$.

Here, if m is a fuzzy subgroup of G , then m_t is called a level subgroup of m .

Definition 2.5 ([1]**) :** *A* fuzzy subset m of a ring R is called a fuzzy left (right) ideal of R if
(i) $m(x-y) \ge \min(m(x), m(y))$
(ii) for all $x, y \in R$ $m(xy) \ge m(y)$ $(m(xy) \ge m(x))$.

A fuzzy subset m of R is called a fuzzy ideal of R if m is a fuzzy left and fuzzy right ideal of R.

Definition 2.6 ([5] **) :** If m is a fuzzy subset of R, then for any $t \in \mathrm{Im}\, m$, the set $m_t = \{x \in R \mid m(x) \geq t\}$ is called the level subset of R with respect to m.

Theorem 2.7 [1]**:** Let m be fuzzy subset of R. m is a fuzzy ideal of R if and only if m_t is an ideal of R for $\forall t \in \mathrm{Im}\, m$.

Here, if m is a fuzzy ideal of R, then m_t is called a level ideal of m.

Definition 2.8 ([6] **) :** A fuzzy relation m on R is the fuzzy subset of $R \times R$.

Definition 2.9 ([3] **):** Let m and s be fuzzy subsets of R. The Cartesian product of m and s is $m \times s\,(x, y) = \min(m(x), s\,(y))$ for all $x, y \in R.$

3. FUZZY SUBGROUPS AND FUZZY IDEALS

Now we will generalize some theorems in [3]..

Theorem 3.1: If m_1 and m_2 are fuzzy subgroups of G_1 and G_2 respectively, then $m_1 \times m_2$ is a fuzzy subgroup of $G_1 \times G_2$.

Proof: Let $(a_1, b_1), (a_2, b_2) \in G_1 \times G_2$.

$$\begin{aligned} m_1 \times m_2((a_1, b_1).(a_2, b_2)) &= m_1 \times m_2(a_1 a_2, b_1 b_2) \\ &= \min(m_1(a_1 a_2), m_2(b_1 b_2)) \\ &\geq \min(m_1(a_1), m_1(a_2), m_2(b_1), m_2(b_2)) \\ &\geq \min(\min(m_1(a_1), m_2(b_1)), \min(m_1(a_2), m_2(b_2))) \\ &= \min(m_1 \times m_2(a_1, b_1), m_1 \times m_2\,(a_2, b_2)) \end{aligned}$$

and

$$\begin{aligned} m_1 \times m_2((a_1, b_1)^{-1}) &= m_1 \times m_2(a_1^{-1}, b_1^{-1}) \\ &= \min(m_1(a_1^{-1}), m_2(b_1^{-1})) \\ &\geq \min(m_1(a_1), m_2(b_1)) \\ &= m_1 \times m_2\ a_1\ b_1 \end{aligned}$$

Therefore $m_1 \times m_2$ is a fuzzy subgroup of $G_1 \times G_2$.

Theorem 3.2: If m_1, m_2 are fuzzy ideals of R_1, R_2 respectively , then $m_1 \times m_2$ is fuzzy ideal of $R_1 \times R_2$.

Proof: $m_1 \times m_2(0_1, 0_2) = \min(m_1(0_1), m_2(0_2))$. Let $t \in \mathrm{Im}(m_1 \times m_2)$ then $t \leq m_1(0_1)$ and $t \leq m(0_2)$. Thus m_{1_t} and m_{2_t} are ideals of R_1 and R_2 respectively. Hence for all $t \in \mathrm{Im}(m_1 \times m_2)$, $(m_1 \times m_2)_t = m_{1_t} \times m_{2_t}$ is left ideal of $R_1 \times R_2$. Because $\forall (x, y), (z, t) \in (m_1 \times m_2)_t$ and $\forall (a, b) \in (R_1, R_2)$ we must show that $(x - z, y - t) \in (m_1 \times m_2)_t$ and $(xa, yb) \in (m_1 \times m_2)_t$. $m_1 \times m_2(x - z, y - t) = \min(m_1(x - z), m_2(y - t))$ and since m_{1_t} and m_{2_t} are ideals of R_1 and R_2 respectively $\min(m_1(x - z), m_2(y - t)) \geq t$ then $(x - z, y - t) \in (m_1 \times m_2)_t$. Since $m_1 \times m_2\ (xa, yb) = \min(m_1(xa), m_2(yb))$ and m_{1_t} and m_{2_t} are ideals of $R_1 \times R_2$ $\min(m_1(xa), m_2(yb)) \geq t$ then $(xa, yb) \in$ $(m_1 \times m_2)_t$. Hence $(m_1 \times m_2)_t$ is ideal of $R_1 \times R_2$.

Corollary 3.3 i) If $m_1, m_2, m_3, ..., m_n$ are fuzzy subgroups of $G_1, G_2, ..., G_n$ respectively , then $m_1 \times m_2 \times m_3 \times ... \times m_n$ is fuzzy subgroups of $G_1 \times G_2 \times ... \times G_n$.

ii) If $m_1, m_2, m_3, ..., m_n$ are fuzzy ideals of $R_1, R_2, ..., R_n$ respectively , then $m_1 \times m_2 \times m_3 \times ... \times m_n$ is fuzzy ideal of $R_1, R_2, ..., R_n$.

Proof: . One can easily show by induction method.

Theorem 3.4: Let m_1, m_2 be fuzzy subsets of G_1, G_2 respectively such that $m_1 \times m_2$ is a fuzzy subgroup of $G_1 \times G_2$. Then m_1 or m_2 is fuzzy subgroup of G_1 or G_2 respectively.

Proof: We know that $m_1 \times m_2(e_1, e_2) = \min(m_1(e_1), m_2(e_2)) \geq m_1 \times m_2(x, y)$, $\forall(x, y) \in G_1 \times G_2$.

Then $m_1(x) \leq m_1(e_1)$ or $m_2(y) \leq m_2(e_2)$. If $m_1(x) \leq m_1(e_1)$, then $m_1(x) \leq m_2(e_2)$ or $m_2(y) \leq m_2(e_2)$. Let $m_1(x) \leq m_2(e_2)$. Then $\forall x \in G_1$ $m_1 \times m_2(x, e_2) = m_1(x)$. $\forall x, y \in G_1$

$$
\begin{aligned}
m_1(xy) &= m_1 \times m_2(xy, e_2) \\
&= m_1 \times m_2((x, e_2)(y, e_2)) \\
&\geq \min(m_1 \times m_2(x, e_2), m_1 \times m_2(y, e_2)) \\
&= \min(m_1(x), m_1(y))
\end{aligned}
$$

and

$$
\begin{aligned}
m_1(x^{-1}) &= m_1 \times m_2(x^{-1}, e_2) \\
&= m_1 \times m_2(x^{-1}, e_2^{-1}) \\
&= m_1 \times m_2(x, e_2)^{-1} \\
&\geq \min m_1 \times m_2(x, e_2) \\
&= m_1(x).
\end{aligned}
$$

Therefore m_1 is fuzzy subgroup of G_1 .

Now suppose that $m_1(x) \leq m_2(e_2)$ is not true for all $x_1 \in G_1$. If $m_1(x) > m_2(e_2)$ $\exists x \in G_1$, then $m_2(y) \leq m_2(e_2)$ $\forall y \in G_2$. Therefore $m_1 \times m_2(e_1, y) = m_2(y)$ for all $y \in G_2$. Similarly $\forall x, y \in G_2$

$$
\begin{aligned}
m_2(xy) &= m_1 \times m_2(e_1, xy) \\
&= m_1 \times m_2((e_1, x)(e_1, y)) \\
&\geq \min(m_1 \times m_2(e_1, x), m_1 \times m_2(e_1, y)) \\
&= \min(m_2(x), m_2(y))
\end{aligned}
$$

and

$$
\begin{aligned}
m_2(x^{-1}) &= m_1 \times m_2(e_1, x^{-1}) \\
&= m_1 \times m_2(e_1^{-1}, x^{-1}) \\
&= m_1 \times m_2(e_1, x)^{-1} \\
&\geq \min m_1 \times m_2(e_1, x) \\
&= m_2(x).
\end{aligned}
$$

Hence m_2 is fuzzy subgroup of G_2 . Consequently either m_1 or m_2 is fuzzy subgroup of G_1 or G_2 respectively.

Theorem 3.5: Let m_1, m_2 be fuzzy subsets of R_1, R_2 respectively such that $m_1 \times m_2$ is a fuzzy ideal of $R_1 \times R_2$. Then m_1 or m_2 is fuzzy ideal of R_1 or R_2 respectively.

Proof: We know that $m_1 \times m_2(0_1, 0_2) = \min(m_1(0_1), m_2(0_2)) \geq m_1 \times m_2(x, y)$, $\forall (x, y) \in R_1 \times R_2$. Then $m_1(x) \leq m_1(0_1)$ or $m_2(y) \leq m_2(0_2)$. If $m_1(x) \leq m_1(0_1)$, then $m_1(x) \leq m_2(0_2)$ or $m_2(y) \leq m_2(0_2)$. Let $m_1(x) \leq m_2(0_2)$. Then $\forall x \in R_1$ $m_1 \times m_2(x, 0_2) = m_1(x)$. $\forall x, y \in R_1$

$$\begin{aligned} m_1(x-y) &= m_1 \times m_2(x-y, 0_2) \\ &= m_1 \times m_2((x, 0_2)(y, 0_2)) \\ &\geq \min(m_1 \times m_2(x, 0_2), m_1 \times m_2(y, 0_2)) \\ &= \min(m_1(x), m_1(y)) \end{aligned}$$

and

$$\begin{aligned} m_1(xy) &= m_1 \times m_2(xy, 0_2) \\ &= m_1 \times m_2((x, 0_2).(y, 0_2)) \\ &= m_1 \times m_2(x, 0_2) \quad \text{or} \quad m_1 \times m_2(y, 0_2) \\ &\geq \min m_1 \times m_2(x, 0_2) \ \text{or} \ \geq \min m_1 \times m_2(y, 0_2) \\ &= m_1(x) \quad \text{or} \quad = m_1(x) \end{aligned}$$

Therefore m_1 is fuzzy ideal of R_1.

Now suppose that $m_1(x) \leq m_2(0_2)$ is not true for all $x_1 \in R_1$. If $m_1(x) > m_2(0_2)$ $\exists x \in R_1$, then $m_2(y) \leq m_2(0_2)$ $\forall y \in R_2$. Therefore $m_1 \times m_2(0_1, y) = m_2(y)$ for all $y \in G_2$. Similarly $\forall x, y \in R_2$

$$\begin{aligned} m_2(x-y) &= m_1 \times m_2(0_1, x-y)) \\ &= m_1 \times m_2((0_1, x) - (0_1, y)) \\ &\geq \min(m_1 \times m_2(0_1, x), m_1 \times m_2(0_1, y)) \\ &= \min(m_2(x), m_2(y)) \end{aligned}$$

and

$$\begin{aligned} m_2(xy) &= m_1 \times m_2(0_1, xy) \\ &= m_1 \times m_2((0_1, x).(0_1, y)) \\ &\geq m_1 \times m_2(0_1, x), \ (m_1 \times m_2(0_1, y)) \\ &= \min(m_1(0_1), m_2(x)) \ ,(= \min(m_1(0_1), m_2(y)) \\ &= m_2(x), \ (= m_2(y)). \end{aligned}$$

Therefore m_2 is fuzzy ideal of R_2.

Corollary 3.6: Let $m_1, m_2, m_3, ..., m_n$ be a similar fuzzy subsets of $G_1, G_2, ..., G_n$ ($R_1, R_2, ..., R_n$) of such that $m_1 \times m_2 \times m_3 \times ... \times m_n$ is fuzzy subgroup (ideal) of $G_1 \times G_2 \times ... \times G_n$ ($R_1, R_2, ..., R_n$). Then m_1 or m_2 or m_3 or ...or m_n is a fuzzy subgroups (ideals) of $G_1, G_2, ..., G_n$ ($R_1, R_2, ..., R_n$) respectively.

Corollary 3.7: Let m_1 and m_2 be a similar fuzzy subsets of G_1 and G_2 (R_1 and R_2) of such that $m_1 \times m_2$ is fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). If $\forall x \in G_1, \forall y \in G_2$ $m_1(e_1) = m_2(e_2)$ $m_1(x) \leq m_1(e_1)$ and ($\forall x \in R_1, \forall y \in R_2$ $m_1(0_1) = m_2(0_2)$, $m_1(x) \leq m_1(0_1)$ and $m_2(y) \leq m_2(0_2)$) then m_1, m_2 is a fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2).

Corollary 3.8: Let $m_1,\ m_2,\ m_3,...,m_n$ be a similar fuzzy subsets of $G_1,\ G_2,...,G_n$ ($R_1,\ R_2,...,R_n$) of such that $m_1 \times m_2 \times m_3 \times ... \times m_n$ is fuzzy subgroup (ideal) of $G_1 \times G_2 \times ... \times G_n$ ($R_1 \times R_2 \times ... \times R_n$). If $\forall x_1 \in G_1, \forall x_2 \in G_2,..., \forall x_n \in G_n$ $m_1(e_1) = m_2(e_2) = m_3(e_3) = ... = m_n(e_n)$, $G_1,\ G_2,...,G_n$ $m_1(x_1) \le m_1(e_1)$, $m_2(x_2) \le m_2(e_2)$, $m_3(x_3) \le m_3(e_3),...,\ m_n(x_n) \le m_n(e_n)$ ($\forall x_1 \in R_1, \forall x_2 \in R_2,..., \forall x_n \in R_n$ $m_1(0_1) = m_2(0_2) = m_3(0_3) = ... = m_n(0_n)$, $m_1(x_1) \le m_1(0_1)$, $m_2(x_2) \le m_2(0_2)$, $m_3(x_3) \le m_3(0_3),...,\ m_n(x_n) \le m_n(0_n)$) then $m_1,\ m_2,\ m_3,...,m_n$ is a fuzzy subgroups (ideals) of $G_1,\ G_2,...,G_n$ ($R_1,\ R_2,...,R_n$) respectively.

4. CONCLUSIONS

One can examine these theorems in any Rings. That is, it true that these theorems are valid in non commutative rings without identity element.

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