THREE DIFFERENT TYPE DIFFERENTIAL QUADRATURE METHODS (DQM) FOR LINEAR BUCKLING ANALYSIS OF UNIFORM ELASTIC COLUMNS

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ÜNİFORM ELASTİK KOLONLARIN DOĞRUSAL BURKULMA ANALİZİ İÇİN ÜÇ FARKLI TİP DİFERANSİYEL QUADRATURE METODU (DQM)

ÖZET

Kolonların burkulma analizi; Diferansiyel Quadrature (DQ), Genelleştirilmiş Diferansiyel Quadrature(GDQ) ve Harmonik Diferansiyel Quadrature (HDQ) metotlarıyla geliştirilmiştir. Diferansiyel quadrature metodu; koordinat doğrultusuna göre bir fonksiyonun türevi, çözüm bölgesi için tanımlanmış bir polinom yardımıyla yaklaşım kurabilen sürekli bir fonksiyon ve o doğrultu boyunca önceden seçilen noktalardaki fonksiyon değerlerinin ağırlıklı bir toplamı olarak ifade edilebileceği prensibine dayanır. Bu prensibin kolon diferansiyel denklemi ve sınır koşullarına uygulanması ile standart bir özdeğer problemi elde edilir. Sayısal uygulamalarda; Ankastre-Ankastre (C-C), Mafsallı- Mafsallı (H-H), Ankastre-Serbest (C-F) ve Ankastre-Mafsallı (C-H) olarak farklı sınır şartlarına sahip dört kolon türü göz önüne alınmıştır. Kullanılan metotların yeterliliğini göstermek için sayısal sonuçlar sunulmuştur.
Anahtar Sözcükler: Diferansiyel quadrature, Elastik kolonlar, Burkulma

ABSTRACT

Differential Quadrature (DQ), Generalized Differential Quadrature (GDQ), and Harmonic differential quadrature (HDQ) methods are developed for the buckling analysis of columns. In the method of differential quadrature, partial space derivatives of a function appearing in a differential equation are approximated by means of a polynomial expressed as the weighted linear sum of the function values at a preselected grid of discrete points. Applying this concept to the governing differential equation and boundary conditions of columns gives a standard eigenvalue equation. Four different columns such as Clamped-Clamped (C-C), Hinged-Hinged (H-H), Clamped-Free (C-F), and Clamped-Hinged (C-H) are taken into consideration in numerical applications. Numerical results are presented to demonstrate its efficiency.
Keywords: Differential quadrature, Elastic columns, Buckling

1. INTRODUCTION

It is well known that, the analysis of engineering systems is included two main stage, such as construction of a mathematical model for a given physical phenomena and the solution of this mathematical equation. With the modern computer technology, various numerical methods were well developed and widely used to solve various kinds of engineering and science problems which are described by the differential equations. This equations either linear or nonlinear and in most cases, their closed form solutions are extremely difficult to establish. As a result, approximate numerical methods have been widely used to solve such a differential equations. The most commonly used numerical methods for such applications are the finite element and finite

difference. Most engineering problems can be solved by these methods for satisfactory accuracy if a proper and sufficient number of grid points are used. However, in a large number of practical applications where only reasonably accurate solutions at few specified physical coordinates are of interest, the finite element or finite difference method becomes inappropriate since they still require a large number of grid points and so large a computer capacity [6,7]. Consequently, both CPU time and storage requirements are often considerable for the standard methods. In seeking a more efficient numerical method which requires fewer grid points yet achieves acceptable accuracy, the method of differential quadrature (DQ) which is based on the assumptions that the partial derivatives of a function in one direction can be expressed as a linear combination of the function values at all mesh points along that direction, was introduced by Bellman et al. [1]. The method of differential quadrature circumvents the above difficulties by computing a moderately accurate solution from only a few points. In this study, buckling analysis of elastic columns having different support conditions is investigated by using DQ, GDQ, and HDQ. The accuracy, efficiency and convenience of HDQ are demonstrated throughout the numerical examples.

2. DIFFERENTIAL QUADRATURE METHOD (DQM)

The idea of the differential quadrature method is to quickly compute the derivative of a function at any grid point within its bounded domain by estimating a weighted linear sum of values of the function at a small set of points related to the domain. As in the case of the other numerical analysis techniques, such as finite element or finite difference methods, the DQM also transforms the given differential equation into a set of analogous algebraic equations in terms of the unknown function values at the reselected sampling points in the domain. The problem areas in which the applications of differential quadrature method may be found in the available literature include static and dynamic structural mechanics and stability analysis of structures [8,10,11,16]. During recent years, the DQM has been largely promoted by Bert and associates who were the first to introduce the method as a tool for structural analysis. In fact, Bert and his associates had been made most of effective supplement to the theory and application on DQM. Recent works of Bert and associates, mainly on the vibration analysis of plates, have contributed significantly to the development of the DQM [3,4]. It has been claimed that the DQM has the capability of producing highly accurate solutions with minimal computational effort. All this work has demonstrated that the application of the DQ methods leads to accurate results with less computational effort and that there is a potential that the method may be become an alternative to the conventional methods such as finite differences and finite element. Therefore research on extension and application of the method becomes an important endeavor.   In the differential quadrature method, a partial derivative of a function with respect to a space variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. For simplicity, we consider a one-dimensional function $\psi(x)$ in the [-1,1] domain, and N discrete points. Then the first derivatives at point i, at $x = x_i$ is given by

$$\Psi_x(x_i) = \frac{\partial \Psi}{\partial x}\bigg|_{x=x_i} = \sum_{j=1}^{N} A_{ij}\, \Psi((x_j)); \qquad i = 1,2,........,N \qquad (1)$$

where x_j are the discrete points in the variable domain, $\psi(x_j)$ are the function values at these points and A_{ij} are the weighting coefficients for the first order derivative attached to these function values. Bellman et al. [2] suggested two methods to determine the weighting coefficients. The first one is to let equation (1) be exact for the test functions

$$\Psi_k(x) = x^{k-1}, \qquad k = 1,2,...,N \qquad (2)$$

which leads to a set of linear algebraic equations

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij}x_j^{k-1}; \text{ for } i = 1,2,.........,N \text{ and } k = 1,2,.........,N \tag{3}$$

which represents N sets of N linear algebraic equations. This equation system has a unique solution because its matrix is of Vandermonde form. This equation may be solved for the weighting coefficients analytically using the Hamming's method [12] or numerical method using the certain special algorithms for Vandermonde equations, such as the method of Bjorck and Pereyra [5]. As similar to the first order, the second order derivative can be written as

$$\Psi_{xx}(x_i) = \frac{\partial^2 \Psi}{\partial x^2}\bigg|_{x=x_i} = \sum_{j=1}^{N} B_{ij}\,\Psi(x_j); \qquad i = 1,2,.........,N \tag{4}$$

where the B_{ij} is the weighting coefficients for the second order derivative. Equation (4) also can be written

$$\Psi_{xx}(x_i) = \frac{\partial^2 \Psi}{\partial x^2}\bigg|_{x=x_i} = \sum_{j=1}^{N} A_{ij} \sum_{k=1}^{N} A_{jk}\,\Psi(x_k); \quad i = 1,2,.........,N \tag{5}$$

Again, the function given by equation (2) is used so that the second order derivative is

$$(k-1)(k-2)x_i^{k-3} = \sum_{j=1}^{N} B_{ij}x_j^{k-1} \tag{6}$$

This can be solved in the same manner as indicated for equation (3) above.

3. GENERALIZED DIFFERENTIAL QUADRATURE (GDQ)

Despite the increasing application of the DQ method in structural analysis, a draw back regarding its ill conditioning of the weighting coefficients with increasing number of grid points used as well as the increasing order of derivatives was pointed out by Civan and Sliepcevich [18]. One way of overcoming this drawback was recently addressed by Shu and Richards [15]. They introduced a recurrence relationship that is used to generate weighting coefficients for any order derivatives from its first-order weighting coefficient. These researchers had been called the generalized differential quadrature of their approximate. Using the simplified and generalized differential quadrature method, the weighting coefficients can be determined as follows:

$$A_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}; \qquad i,j = 1,2,........,N, \text{ but } j \neq i \tag{7}$$

where

$$M^{(1)}(x_i) = \prod_{j=1, j \neq i}^{N_x} (x_i - x_j) \tag{8}$$

and

$$A_{ij}^{(r)} = r\,[A_{ii}^{(r-1)}A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{x_i - x_j}]; \quad \text{for } i,j = 1,2,....,N, \text{ but } j \neq i; \text{ and}$$

$$r = 2,3,....., N\text{-}1 \tag{9}$$

$$A_{ii}^{(r)} = - \sum_{j=1, j \neq i}^{N} A_{ij}^{(r)}, \quad i = 1,2,....,N \text{ and } r = 1,2,.....,N-1 \tag{10}$$

4. HARMONIC DIFFERENTIAL QUADRATURE (HDQ)

In the past ten years, many researchers had been interested the various type DQ method. These are known as the DQ method, Generalized Differential Quadrature (GDQ), and Quadrature Elements Method (QEM). A recently approach the original differential quadrature approximation called the Harmonic differential quadrature (HDQ) has been proposed by Striz et al.[17]. Unlike the differential quadrature that uses the polynomial functions, such as Lagrange interpolated, and Legendre polynomials as the test functions, harmonic differential quadrature uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. The harmonic test function $h_k(x)$ used in the HDQ method is defined as [14];

$$h_k(x) = \frac{\sin\frac{(x-x_0)\pi}{2} \cdots \sin\frac{(x-x_{k-1})\pi}{2} \sin\frac{(x-x_{k+1})\pi}{2} \cdots \sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2} \cdots \sin\frac{(x_k-x_{k-1})\pi}{2} \sin\frac{(x_k-x_{k+1})\pi}{2} \cdots \sin\frac{(x_k-x_N)\pi}{2}}$$

For $k = 0,1,2,....,N$ \tag{11}

According to the HDQ, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i - x_j)/2]\pi}, \qquad i,j = 1,2,3,...,N \tag{12}$$

where

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i - x_j)/2]\pi}, \qquad i,j = 1,2,3,...,N \tag{13}$$

The weighting coefficients of the second-order derivatives B_{ij} for $i \neq j$ can be obtained by using following formula:

$$B_{ij} = A_{ij}\left[2A_{ii}^{(1)} - \pi ctg\left(\frac{x_i - x_j}{2}\right)\pi\right], \qquad i,j = 1,2,3,...,N \tag{14}$$

The weighting coefficients of the first-order and second-order derivatives $A_{ij}^{(p)}$ for $i = j$ are given as

$$A_{ii}^{(p)} = - \sum_{j=1, j \neq i}^{N} A_{ij}^{(p)}, p = 1 \text{ or } 2; \text{ and for } i = 1,2,..., N \tag{15}$$

The weighting coefficient of the fourth order derivative can be computed easily from B_{ij} by [14]

$$D_{ij} = \sum_{k=1}^{N} B_{ik}B_{kj} \tag{16}$$

The main advantage of HDQ over the differential quadrature (DQ) is its ease of the computation of the weighting coefficients without any restriction on the choice of grid points. A factor decisive to the accuracy of the all type differential quadrature solution is the choice of the sampling or grid points. It should be mentioned that in the differential quadrature solutions, the sampling points in the various coordinate directions might be different in number as well as in their type. A

convenient choice for sampling points is that of equally spaced point. This type sampling points are given as

$$x_i = \frac{i-1}{N-1} ; \qquad i = 1,2,.....,N \qquad (17)$$

in the related direction.

5. NUMERICAL APPLICATIONS AND RESULTS

Columns subjected to different boundary conditions are selected as the test examples to demonstrate the applicability and accuracy of the methods. The governing differential equation of column is presented. The present formulation is based on classical small deflection theory. Then, the differential quadrature methods have been applied to this differential equation. Results are obtained for each case using various numbers of grid points. It is observed that the convergence of the methods is very good. Reasonably, accurate results can be achieved by using 7 and 9 points. The computational time on a standard PC (Pentium II having 64 RAM) is less than 1sec for all the cases.

5.1. Buckling of Linear Elastic Columns

The non-dimensional governing differential equation for buckling behavior of an elastic column is given by [19];

$$EI\frac{d^4w}{dX^4} + PL^2\frac{d^2w}{dX^2} = 0 \qquad (18)$$

Differential quadrature form of this equation is

$$\sum_{j=1}^{N} D_{ij} W_j + \lambda \sum_{j=1}^{N} B_{ij} W_j = 0 \; ; \quad i = 1, 2, 3, ...,N \qquad (19)$$

where X= x/L, $\lambda = PL^2/EI$. Numerical applications have been done for a linearly elastic column under four sets of different boundary conditions, namely hinged-hinged (H-H), clamped-hinged (C-H), clamped-clamped (C-C), and clamped-free (C-F). Following, boundary conditions are given for these four cases, respectively.

Clamped-Hinged (C-H)
W= 0 and (dW/dX) =0 at X = 0
W = 0 and (d^2W/dX^2)=0 at X = 1
Clamped-Clamped supported (C-C)
W = 0 and dW/dX =0 at X = 0
W= 0 and dW/dX =0 at X = 1
Hinged-Hinged (H-H)
W = 0 and $d^2 W/dX^2$ =0 at X = 0
W = 0 and $d^2 W/dX^2$ =0 at X = 1
Clamped supported-free end (C-F)
W = 0 and (dW/dX) =0 at X = 0
(d^2W / dX^2 = 0) and (d^3W / dX^3 = 0) at X =1

Applying the differential quadrature approximation to the above equations at each discrete point on the grid. For case (C-H) support conditions can be written as follows:

$$W_1 = 0 \;\; \text{and} \;\; \sum_{j=1}^{N} A_{1j} W_j = 0 ; \; W_N = 0 \;\; \text{and} \;\; \sum_{j=1}^{N} B_{Nj} W_j = 0 \qquad (20)$$

For (C-C):

$$W_1 = 0 \text{ and } \sum_{j=1}^{N} A_{1j} W_j = 0 \text{ ; } W_N = 0 \text{ and } \sum_{j=1}^{N} A_{Nj} W_j = 0 \qquad (21)$$

For (H-H):

$$W_1 = 0 \text{ and } \sum_{j=1}^{N} B_{1j} W_j = 0 \text{ ; } W_N = 0 \text{ and } \sum_{j=1}^{N} B_{Nj} W_j = 0 \qquad (22)$$

and (C-F):

$$W_1 = 0 \text{ and } \sum_{j=1}^{N} A_{1j} W_j = 0 \text{ ; } \sum_{j=1}^{N} B_{Nj} W_j = 0 \text{ and } \sum_{j=1}^{N} C_{Nj} W_j = 0 \qquad (23)$$

The set of equations (19), (20),(21),(22) and (23) or other equations for beam having different boundary conditions are redundant. In order to remove this redundancy, we have to drop the certain equations such as; $i = 1, 2,(N-1)$ and (N) in the essence equation (19). Consequently, Eq. (19) can be rewrite as,

$$\sum_{j=1}^{N} D_{ij} W_j + \lambda \sum_{j=1}^{N} B_{ij} W_j = 0 \qquad \text{for } i = 3,4,\ldots\ldots,(N-2) \qquad (24)$$

Thus, the buckling load of column under a given axial load P can be found by solving above eigenvalue equations with appropriate boundary condition. Table 1 summarizes numerical results of buckling loads by HDQ for case of linear elastic columns with four different boundary conditions. It is shown that in this table, HDQ results using five grid points is more than accurate than the DQ for seven grid points. The buckling load by Chajes [20] using the finite difference method is also presented in Table 1 for comparison. For the buckling load of columns, N=9 grid points provide acceptable results with a maximum discrepancy of 0.005% for the clamped-clamped (C-C) boundary conditions and a maximum discrepancy of 0.04% for clamped-free (C-F) support case for HDQ. In addition to this, if we use the DQ method with N=9 grid points provide acceptable results with a maximum discrepancy of 0.17 for the C-C support condition and 0.25% for C-F support condition. The obtained buckling loads are compared with those calculated the DQ, GDQ and the exact method. The harmonic differential quadrature results are generally in agreement with the results produced from the analytical [19] and the finite differences [20] results. It is found that the HDQ method possesses both the advantages of DQ and the flexibility of the FD. It is also concluded that the HDQ method contains both the advantages of DQM and the flexibility of the GDQ.

Table 1. Comparison of buckling loads for different columns

	FD Ref.20 (N=5)	Exact (Ref.19)	HDQ (N=5)	HDQ (N=7)	HDQ (N=9)	DQ (N=9)	GDQ (N=9)
C-C	41.360	39.478	38.745	39.034	39.476	40.177	39.450
C-F	2.333	2.467	2.556	2.304	2.466	2.289	2.464
H-H	11.548	9.869	10.184	9.621	9.869	9.774	9.872
C-H	22.296	20.142	22.960	21.025	20.142	20.919	20.204

The variation of the error with the number of grid points was shown in Fig.1 for the various type differential quadrature methods. The percentage error had been reduced as parallel to the increase the grid points. In this figure (Fig. 1), H-H supports case is taken as boundary conditions. The best solution is obtained for N= 11 grid points by using HDQ method.
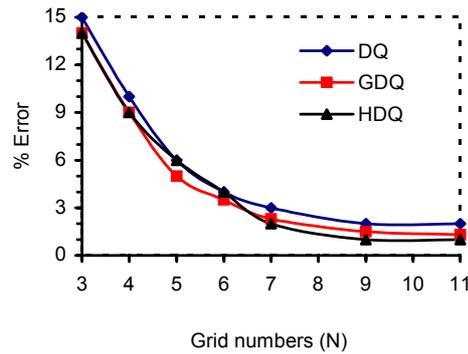
Figure 1. Percentage error with grid numbers

6. CONCLUDING REMARKS

DQ is recently proposed and there are only a few papers on this novel kind numerical method. Three different type DQ methods were introduced to study the buckling analysis of elastic columns. The method of generalized and harmonic differential quadrature that was using the paper proposes a very simple algebraic formula to determine the connections weighting coefficients required by DQ approximation without restricting the choice of mesh grids. The known boundary conditions are easily incorporated in the HDQ as well as the other type differential quadrature. The discretizing and programming procedures are straightforward and easy.

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