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ARAŞTIRMA MAKALESİ

ON THE MEAN INTEGRITY OF A GRAPH

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ÖZET

Bir iletişim ağının **Zedelenebilirlik değeri,** bağlantı hatlarının veya istasyonların başarısız olmasından sonra operasyonun kesilmesine göre ağın dayanma gücünü ölçer. Zedelenebilirliği ölçmek için, graf teorisindeki parametrelerden birisi **ortalama bütünlük**tür. G bir graf ve $S \subset V(G)$ olsun. G grafından S kümesinin elemanları silindiğinde kalan graf G – S ile gösterilir. G – S grafı en az bir bileşen içerir ve G – S nin bileşenlerinin her biri $p_1, p_2, \ldots, p_k$ tepeli ise $\overline{m}(G - S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i$ olmak üzere bir G grafının **ortalama bütünlüğü** , J(G) ile gösterilmiş ve

$$J(G) = \min_{S \subset V(G)} \{ |S| + \overline{m}(G - S) \}$$

şeklinde ifade edilmiştir.

Bu makale **ortalama bütünlük** için sınırlar ve ortalama bütünlük ile diğer graf parametreleri arasındaki bağıntılar için sonuçlar içerir.

SUMMARY

In a communication network, the **vulnerability** measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. In order to measure the vulnerability, one of the parameters in graph theory is **mean integrity.** Let G be a graph of order p and S be a subset of V(G). When the elements of S are deleted from G, the remaining graph is denoted by G – S. The graph G – S contains at least a component and if the each one of the components of G – S have orders $p_1, p_2, \ldots, p_k$, then $\overline{m}(G - S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i$.

Formally, the **mean integrity** of a graph G, **denoted J(G)**, is defined as

$$J(G) = \min_{S \subset V(G)} \{ |S| + \overline{m}(G - S) \}.$$

This paper contains results on bounds for the mean integrity, on relationships between mean integrity and other graph parameters.

1. INTRODUCTION

In a communication network, the **vulnerability** measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a graph as a modelling a network, the stations of network correspond to vertices of the graph and the communication links of network correspond to edges of the graph. The analysis of vulnerability in networks generally involves some questions about how the underlying graph is connected. When some vertices of a graph (the stations of network) are deleted, one wants to know whether the remaining graph is still connected. Moreover if the graph is disconnected, the determination of the number of its components or their orders (the number of vertices of components) is useful. In order to measure the vulnerability, one of the most studied and best known parameters in graph theory is **mean integrity**. To make the development of this parameter, we give the definitions step by step.

The order of a graph G (that is, the number of vertices) is denoted by p. As usual V(G) and E(G) will denote respectively the sets of vertices and edges of G. Let G be a graph of order p and S be a subset of V(G). When the elements (vertices) of S are deleted from G, the remaining graph is denoted by G – S. The graph G – S contains at least one component and if the each one of the components of G – S have orders p_1, $p_2,\ldots,p_k$, then $\overline{m}(G-S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i$. Formally, the **mean integrity** of a graph G, **denoted J(G)**, is defined as

$$J(G) = \min_{S \subset V(G)} \{ |S| + \overline{m}(G-S) \}.$$

It was introduced as a measure of graph vulnerability by Chartrand, G., Kapoor, S.F., McKee, T.A. and Oellermann, O.R. **[4]**.

Next we will give an example for the mean integrity.

Question: Let P_3 be a path graph of order 3 (Figure 1). What is the mean integrity of the graph P_3?

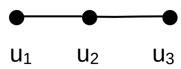


Figure 1

Answer: For the answer, we have the following cases:

Case 1: If S = $\{u_1\}$ or S = $\{u_3\}$, then graph G – S have a component of order 2. Hence we have

$$\overline{m}(G - S) = \frac{2^2}{2} = 2 \text{ and } |S| = 1 \tag{1}$$

Case 2: If S = $\{u_2\}$, then G –S have two components of order 1. Thus we have

$$\overline{m}(G - S) = \frac{1^2 + 1^2}{1+1} = 1 \text{ and } |S| = 1 \tag{2}$$

Case 3: If S = $\{u_1,u_2\}$ or S = $\{u_2,u_3\}$ or S = $\{u_1,u_3\}$, then G – S have only one component of order 1. Hence we have

$$\overline{m}(G - S) = \frac{1^2}{1} = 1 \text{ and } |S| = 2 \tag{3}$$

By using (1), (2) and (3), we have

$$J(P_3) = \min_{S \subset V(P_3)} \{|S| + \overline{m}(P_3 - S)\} = \min\{1+2, 1+1, 2+1\} = 2$$

Now we will give some definitions and theorems. These theorems are obtained by Chartrand et al.

Definition 1: A set of vertices of a graph G is called **J - set** of G if

$$J(G) = |S| + \overline{m}(G - S) = |S| + \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i .$$

Theorem 1: [4] If S is a J - set of a graph G that is not complete, then G – S is disconnected.

Definition 2: The **connectivity** κ (G) of a graph G is the minumum cardinality of a set X of vertices of G such that G – X is either disconnected graph or isolated vertex. A graph G of order p is **q-connected** ($0 \leq q \leq p-1$) if $\kappa(G) \geq q$.

Theorem 2: [4] If G is non-complete then every J – set of G is a cut – set of G and hence has cardinality at least κ (G).

Theorem 3: [4] If G is an q-connected graph, then $J(G) \geq q+1$.

Theorem 4: [4] Let G be a graph of order p and q an integer with $1 \leq q < p$. If

$$J(G) > p - 2 + \frac{2}{p-q+1}, \text{ then G is q-connected.}$$

Definition 3: A subset X of V(G) is called an **independent set** of G if no two vertices of X are adjacent in G. An independent set is maximum if G has no independent set X' with $|X'| > |X|$. The number of vertices in a maximum independent set of G is called the **independence number** of G and is denoted by β **(G)**.

Definition 4: A vertex and an edge are said to cover each other in a graph G if they are incident in G. A vertex **cover** in G is a set of vertices that covers all edges of G. The minumum cardinality of a vertex cover in a graph G is called the **vertex covering number** of G and is denoted by α (G).

Theorem 5: [4] Let δ (G) be minumum vertex degree of a graph G. For every graph G,

$$\delta(G)+1 \leq J(G) \leq \alpha(G)+1.$$

Theorem 6: [5] For any graph G of order p, $\alpha(G)+\beta(G)=p$.

In the next section, we give certain results for the mean integrity of a graph G.

2. RESULTS FOR THE MEAN INTEGRITY

Theorem 7: Let G be a graph of order p. If J(G) = κ (G)+1 iff κ (G) = α(G) .

Proof: ($\Rightarrow$) Let S be an J-set of graph G. Then J(G) = $|S| + \overline{m}(G-S) = \kappa(G)+1$. Moreover we have $|S| \geq \kappa(G)$ by theorem 2. Thus $\kappa(G) + \overline{m}(G-S) \leq \kappa(G)+1 \Rightarrow$ $\overline{m}(G-S) \leq 1$. Since $\overline{m}(G-S) > 0$ for every graph G –S, then $\overline{m}(G-S) = 1$. Hence we have the following statements:

i) If $\overline{m}(G-S) = 1$, then S is a cover set and we have $|S| = \alpha(G)$.

ii) If $J(G) = |S| + \overline{m}(G-S) = \kappa(G)+1$ and $\overline{m}(G-S) = 1$, then $|S| = \kappa(G)$.

Consequently, $\kappa(G) = \alpha(G)$.

($\Leftarrow$) Let S be a set of deleting vertices from G and $|S|$ = a. Then we have the following cases:

i) If $|S| < \kappa(G)$, then we have only one component of order (p-a) and

$\overline{m}(G-S) = \frac{(p-a)^2}{p-a} = p-a$. So

$$J(G) = a + p - a = p \tag{4}$$

ii) Let $|S| = \kappa(G)$. Since $\kappa(G) = \alpha(G)$, graph G -S contains isolated vertices and $\overline{m}(G-S) = 1$. So

$$J(G) = \kappa(G)+1 \tag{5}$$

iii) Let $|S| > \kappa(G)$. If $\kappa(G) = \alpha(G)$, then $\overline{m}(G-S) \geq 1$ and so

$$J(G) \geq \kappa(G)+2 \tag{6}$$

Since $\kappa(G) \leq p-1$ for every graph G, we have by using (4), (5) and (6)

$$J(G) = \min\{p, \kappa(G)+1, \kappa(G)+2\} = \kappa(G)+1. \square$$

Theorem 8: Let G be a graph of order p. Then J(G) = 2 iff $\alpha(G) = 1$.

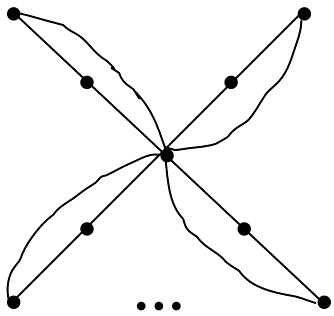
Proof: ($\Rightarrow$) Let S be a J -set of G. Since $J(G) = |S| + \overline{m}(G-S) = 2$ by the hypotesis, we have $|S| = 1$ and $\overline{m}(G-S) = 1$ (Otherwise, if $|S| = 2$, then $\overline{m}(G-S) = 0$. But $\overline{m}$ (G - S) must be at least 1 for every graph G – S). Then $\overline{m}(G-S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i = 1$ and so $|p_i| = 1$ for every i. That is, each one of components of G – S is isolated vertex. Hence S is a cover set and $|S| = \alpha(G) = 1$.

($\Leftarrow$) If we remove only one vertex, then we have the isolated vertices by $\alpha(G) = 1$. Hence $\overline{m}(G-S) = \dfrac{1^2+1^2+\cdots+1^2}{1+1+\cdots+1} = 1$ and $|S| = 1$. Thus J(G) = 1+1= 2. $\square$

Theorem 9: Let G be a graph of order p. Then J(G) = 3 iff $G \cong K_1 + rK_2$ $(r > 0)$ or G is isomorphic to a star graph with diameter 4 or $\alpha(G) = 2$.

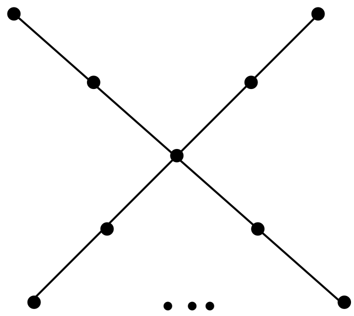
Proof: ($\Rightarrow$) Let S be a J -set of G. Since $J(G) = |S| + \overline{m}(G-S) = 3$, we have two cases:

Case 1: Let $|S| = 1$ and $\overline{m}(G-S) = 2$. Then $\overline{m}(G-S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i = 2$ and we easily seen that $|p_i| = 2$ for every i. That is, each one of components of G - S has exactly two vertices. Hence G must be one of the following graphs.



$K_1 + rK_2$ $(r > 0)$

Figure 2



A star graph of diameter 4

Figure 3

In fact, if we delete the vertex of maximum degree of these graphs, we obtain the components of two vertices. Consequently, $G \cong K_1 + r K_2$ (r > 0) or G is isomorphic to a star graph with diameter 4.

Case 2: Let $|S|=2$ and $\overline{m}(G - S) = 1$. Then $\overline{m}(G - S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i = 1$ and we easily seen that $|p_i|=1$ for every i. That is, we have the isolated vertices and graph G – S has no any edge. Then the set S must be a cover set and $\alpha(G) = |S| = 2$.

($\Leftarrow$) If we delete the vertex of maximum degree from graph $K_1 + r K_2$ (Figure 2), each one of the components p_i has exactly two vertices. Hence

$$\overline{m}(G - S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i = \frac{2^2 + 2^2 + \cdots + 2^2}{2 + 2 + \cdots + 2} = 2 \text{ and } |S| = 1.$$

Consequently, $J(K_1 + r K_2) = 1+2 = 3$.

Similiarly we easiliy seen that if G is isomorphic to a star graph with diameter 4 (Figure 3), then J(G) = 3.

Let $\alpha(G) = 2$ and S be a cover set. Then $|S| = 2$ and if we delete the vertices of S from G, the remaining graph contains only isolated vertices. Hence $\overline{m}(G - S) = 1$ and J(G) = 2+1= 3 . □

Theorem 10: Let G be a non-complete graph of order p. Then J(G) = p -1 iff $\beta(G) \le 2$.

Proof: ($\Rightarrow$) We have $J(G) \le \alpha(G)+1$ and $\alpha(G) + \beta(G) = p$ by theorem 5 and 6, respectively. Then

$$J(G) = p-1 \le \alpha(G)+1 \Rightarrow p \le \alpha(G)+2 \Rightarrow \alpha(G) + \beta(G) \le \alpha(G)+2 \Rightarrow \beta(G) \le 2.$$

($\Leftarrow$) If $\beta(G) \le 2$, then $p - \beta(G) \ge p - 2$ and $\alpha(G) \ge p - 2$ by theorem 6. Since the graph G is not complete graph, $\alpha(G)$ must be p – 2 (Because, $\alpha(G)$ = p-1 iff G is a complete graph of order p). Then if we delete the vertices of cover set from graph G, we have two components of order 1. So $\overline{m}(G - S) = \frac{1^2 + 1^2}{2} = 1$ and $|S| = p - 2$.

Consequently, J(G) = p –2 + 1 = p – 1. □

In the next theorem we give an upper bound involving some graphical parameters by using J(G). Let $X \subseteq V(G)$ and n(G – X) be the maximum order of a component of G -X.

Theorem 11: Let t be a positive integer. If $J(G) = \kappa(G) + t$, then

$$\alpha(G) \le \kappa(G) + (\beta(G)+1)(t-1).$$

Proof: Let S be an J – set of G. Since J(G) = κ(G)+ t, we have $\kappa(G) \leq |S| \leq \kappa(G)+t-1$ and $\overline{m}(G-S) \leq t$. Then $\overline{m}(G-S) = \sum_{i=1}^{k} p_i^2 / \sum_{i=1}^{k} p_i \leq t \Rightarrow \sum_{i=1}^{k} p_i^2 \leq t \sum_{i=1}^{k} p_i$. Hence we easily seen that $|p_i| \leq t$ for every i. That is, the order of each one of the components is at most t. Thus we have $n(G-S) \leq t$. Moreover we know that $n(G-S) \geq \frac{p-|S|}{\beta(G)}$ for every graph G of order p where $S \subset V(G)$ **[6]**. Hence $\frac{p-|S|}{\beta(G)} \leq n(G-S) \leq t \Rightarrow$ $p-|S| \leq t\beta(G)$. Since $\alpha(G)+\beta(G) = p$ by theorem 6, $\alpha(G)+\beta(G)-|S| \leq t\beta(G)$ $\Rightarrow \alpha(G) \leq |S| + (t-1)\beta(G)$. On the other hand, since $|S| \leq \kappa(G)+t-1$ we have

$$\alpha(G) \leq \kappa(G) + (\beta(G)+1)(t-1). \square$$

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